VOLUME 81 PAPER No. 735

PROCEEDINGS

AMERICAN SOCIETY
OF
CIVIL ENGINEERS

JULY, 1955

NATURAL FREQUENCIES OF CONTINUOUS FLEXURAL MEMBERS

by A. S. Veletsos, J.M. ASCE,
and N. M. Newmark, M. ASCE

STRUCTURAL DIVISION

{Discussion open until November 1, 1955}

Copyright 1955 by the AMERICAN SOCIETY OF CIVIL ENGINEERS
Printed in the United States of America

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

THIS PAPER

--represents an effort by the Society to deliver technical data direct from the author to the reader with the greatest possible speed. To this end, it has had none of the usual editing required in more formal publication procedures.

Readers are invited to submit discussion applying to current papers. For this paper the final date on which a discussion should reach the Manager of Technical Publications appears on the front cover.

Those who are planning papers or discussions for "Proceedings" will expedite Division and Committee action measurably by first studying "Publication Procedure for Technical Papers" (Proceedings Paper No. 290). For free copies of this Paper—describing style, content, and format—address the Manager, Technical Publications, ASCE.

Reprints from this publication may be made on condition that the full title of paper, name of author, page reference, and date of publication by the Society are given.

The Society is not responsible for any statement made or opinion expressed in its publications.

This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

NATURAL FREQUENCIES OF CONTINUOUS FLEXURAL MEMBERS

A. S. Veletsos,* J. M. ASCE, and N. M. Newmark,** M. ASCE

SYNOPSIS

A numerical method is presented for calculating the natural frequencies of undamped flexural vibration of continuous beams on rigid supports and of rigid jointed plane frameworks without sidesway. The method is based on concepts which are familiar to structural engineers, and it is reduced to a routine system of numerical calculations which, when repeated a sufficient number of times, will give the natural frequencies of the system considered to any desired degree of accuracy. Numerical values for the various physical constants which are necessary in the analysis by this method are tabulated in the Appendix. The application of the method is illustrated by three numerical examples.

INTRODUCTION

The problem of calculating natural frequencies of vibration of structures is of great practical importance in the analysis and design of structures subjected to time-dependent or dynamic forces. This problem is particularly significant in the study of the behavior of structures under stationary periodic forces, such as those which arise from rotating or reciprocating machinery. If the frequency of the disturbing force is sufficiently close to one of the natural frequencies of the structure, the structure will undergo vibrations which, in the absence of appreciable damping effects, may become prohibitively large.

The purpose of this paper is to describe a method for calculating the natural frequencies of flexural vibration of elastic beams continuous over rigid supports, and of rigid jointed, plane, elastic frameworks without sidesway. The mass of the members composing the structure is assumed to be uniformly distributed along each member. A system with distributed mass and elasticity has an infinite number of natural frequencies. With the method presented herein one is capable of determining as many of the natural frequencies as one desires.

The application of classical analytical methods for determining natural frequencies of continuous beams and frames has been described, among others, by Darnley,(1)1 Timoshenko,(2) Bennon,(3) and Hohenemser and Prager.(4) Additional methods having a fairly general applicability have been presented by Mudrak(5,6) Saibel(7) and Looney,(8),A.M. ASCE, and a numerical method applicable primarily to fundamental frequencies has been developed by Masur.(9)

---

* Research Asst. Prof. of Civ. Eng., Univ. of Illinois, Urbana, Ill.
** Research Prof. of Structural Eng., Univ. of Illinois, Urbana, Ill.

1. Numbers in parentheses, unless otherwise identified, refer to the Bibliography at the end of the paper.

The method presented in this paper is an extension of the well-known Holzer's method[10] for calculating natural frequencies of torsional vibration of shafts. It is based on principles which are familiar to structural engineers, and it is reduced to a straight-forward system of numerical calculations which, when repeated a sufficient number of times, will give any of the natural frequencies of the system considered to any desired degree of accuracy. This procedure is simple, of wide applicability, and appears to be less subject to computational errors than any of the other methods known to the authors.

This paper is divided into three parts and an Appendix. Part I describes the principles underlying the method and gives definitions of the various physical constants entering in the analysis. Numerical values of these constants are tabulated in the Appendix. The application of the method to continuous beams is described in Part II, and the extension of the method to continuous frames is explained in Part III. For convenient reference, a detailed outline is included of the steps involved in the application of the method to particular problems, and the method is illustrated by three numerical examples.

Part I. General Description of Method

Basic Principles

The method is based on the fact that, in the absence of damping, the exciting couple which is necessary to maintain a dynamical system in a steady-state of forced vibration with finite amplitudes becomes equal to zero when the frequency of the couple is equal to one of the natural frequencies of the system. It is assumed that the exciting couple is applied at a joint for which the amplitude of rotation is different from zero for all the natural frequencies that are to be determined.

Consider that a continuous beam or frame is in a steady-state of forced vibration under the action of a harmonically varying exciting couple applied at a suitable joint, designated as joint z. The amplitude of this couple is assumed to be such that the amplitude of the rotation (or of the internal bending moment) at a joint of the structure different from z, say at joint 1, has a prescribed finite value. The vibration of the structure is then harmonic and its frequency is equal to that of the exciting couple. Furthermore, since the effect of damping is neglected, the amplitude of vibration is constant and the response is either in phase with or 180^{o} out of phase with the exciting couple.

For a given system, the magnitude of the exciting couple necessary to maintain the prescribed amplitude of vibration at joint 1 depends on the frequency of the couple. For the limiting case of zero frequency, the magnitude of the couple is obviously finite; if desired, its value may be calculated by any of the available methods of indeterminate stress analysis. As the frequency of the exciting couple increases above zero, the structure is acted upon by distributed inertia forces of increasingly greater magnitudes. These forces are at first in phase with the exciting couple and produce displacements in excess of those produced by the couple acting statically. With a further increase in the frequency of the exciting couple, the inertia effects predominate, and for a frequency equal to the fundamental or lowest natural frequency of the structure, the magnitude of the couple becomes equal to zero. This condition of the vanishing exciting couple is characteristic of all the natural frequencies of an undamped elastic system.

Therefore, to determine the natural frequencies of a structure the following approach may be used: (a) consider that the structure oscillates in a

steady-state at an assumed frequency of vibration and with a fixed amplitude of rotation (or internal bending moment) at joint 1; (b) determine the magnitude of the exciting couple at joint z necessary to maintain the vibration; (c) repeat these steps for a number of assumed frequencies; and (d) plot the magnitude of the exciting couple as a function of the frequency of vibration. The frequencies for which the exciting couple vanishes represent the desired natural frequencies of the structure.

It should be noted that the amplitude of rotation (or internal bending moment) at joint 1 is assumed to be different from zero in this method. Accordingly, if there exists a natural frequency for which this assumption is not satisfied, the method will fail to detect it. Such a condition may occur in the case of continuous frames. A supplementary technique for determining the natural frequencies which the main procedure may fail to detect is described in Part III of the paper.

The primary computational task in the method outlined is the evaluation of the exciting couple. This may be accomplished by a number of different procedures. The conditions to be satisfied are simply those of equilibrium and continuity for each joint of the structure. These may be expressed by a set of linear equations in a number of different ways, and the equations may be solved by a variety of procedures. In the procedure presented herein, these conditions are expressed in the form of a dynamic three-slope equation, and the distortions of the joints are computed by the repeated application of this equation, working progressively from one end of the structure to the other.

Elastic Constants for a Vibrating Bar

The following terms refer to a bar which is simply supported at one end and fixed at the other, and is in a steady state of oscillation under the action of a harmonically varying bending moment applied at the simply supported end.

The amplitude of the bending moment necessary to produce, at the end at which the exciting moment is applied, a steady-state rotation of unit amplitude is defined as the "dynamic flexural stiffness" and will be denoted by the symbol K.

The ratio of the bending moment developed at the far fixed end to the exciting moment at the near end is defined as the "dynamic flexural carry-over factor," and will be denoted by k.

The foregoing terms are generalizations of those introduced by Hardy Cross,[11] Hon. M. ASCE, for the analysis of frames under static loads, and they were first used by R. E. Gaskell[12] in applying the method of moment distribution to the analysis of the steady-state forced vibration of frames.

For certain conditions of symmetry, antisymmetry, and for those cases for which the degree of restraint at the far end of a member is known, it may be convenient to use effective stiffnesses. The effective stiffness for a symmetrically deformed bar is

$$K^s = K(1 - k). \qquad (1)$$

For an antisymmetrically deformed bar, the corresponding stiffness is

$$K^a = K(1 + k). \qquad (2)$$

For a bar, the far end of which is subjected to a rotational elastic restraint of stiffness R, the effective stiffness is

$$K' = K - \frac{(kK)^2}{K + R}. \qquad (3)$$

These expressions are identical to those for the static case.[13,14]

The discussion in this paper will be restricted to members having uniform cross section and mass per unit of length. The stiffness and the carry-over factor for both ends of such a member are equal.

Numerical Values of Constants

The pertinent analytical expressions for the quantities defined in the preceding section can be found in reference (12). The dynamic stiffness may be expressed as

$$K = C_K \frac{EI}{L}, \qquad (4)$$

where C_K is a dimensionless factor depending on the dimensionless parameter

$$\lambda = \sqrt[4]{\frac{m\omega^2}{EI}} L. \qquad (5)$$

In the foregoing expressions, m is the mass per unit of length of the bar; ω is the circular frequency of vibration; E is the modulus of elasticity of the material in the bar; I is the moment of inertia of the cross section of the bar about its centroidal axis; and L is the span length of the bar. The carry-over factor k is dimensionless and depends also on the parameter λ.

A graphical representation of the variation of the stiffness and the carry-over factor as a function of λ is given in Fig. 1. For $\lambda = 0$ these quantities have the well known static values of $K = 4EI/L$ and $k = 0.5$. Values of λ equal to 3.927, 7.069 and 10.210 correspond, respectively, to the first, the second, and the third natural frequencies of a bar fixed at one end and simply supported at the other. For these frequencies, the value of the dynamic stiffness is equal to zero since no exciting moment is required to maintain the vibration. Furthermore, since the amplitude of the moment at the fixed end of the bar is different from zero for these frequencies, the value of the carry-over factor is infinite. Values of λ equal to 4.730 and 7.853 correspond to the first and the second natural frequencies of a bar fixed at both ends. For these frequencies the stiffness of the bar becomes infinite since the end rotations are zero.

In the procedure presented herein, only the stiffness and the product of the stiffness and the carry-over factor are needed. Numerical values of these quantities have been computed to seven significant figures[15] for values of λ between zero and 10.20 at increments of 0.01. A portion of these results are tabulated to five significant figures in the Appendix. It should be noted that some of these values could have been obtained from functions tabulated in reference (4).

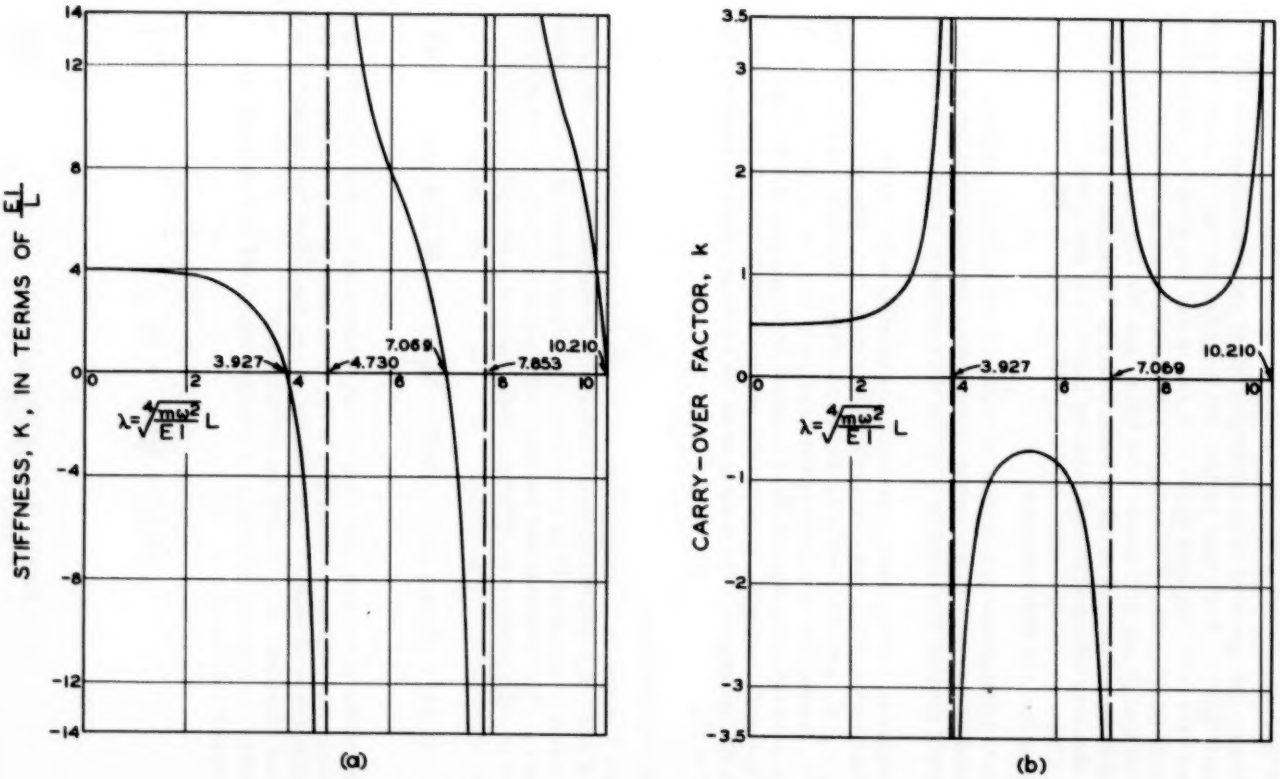


FIG. 1 STIFFNESS K AND CARRY-OVER FACTOR k

Part II. Continuous Beams

Characteristics of Beams Considered

The beams are assumed to be initially straight and continuous over non-deflecting supports. The cross section and the mass per unit of length of each beam may vary from one span to the next, but in any one span these quantities are considered constant. It is assumed that the vibration is restricted to one of the principal planes of flexure of the beam, and that the cross sectional dimensions of each span are small in comparison to its length, so that the effects of shearing deformation and rotatory inertia are negligible.

Sign Convention and Nomenclature

Downward deflections and clockwise rotations are taken as positive. Internal bending moments in a member at its connection with other members, and external couples are taken as positive when acting in a clockwise direction.

The supports are numbered consecutively starting with 1 at the extreme left end and terminating with $\underline{z}$ at the extreme right end. The quantities L_j, E_jI_j, m_j, λ_j, K_j, and $(KK)_j$ refer to the span between supports j and j + 1. The symbol θ_j denotes the amplitude of the rotation at the j-th support and M_j denotes the corresponding amplitude of internal bending moment. The subscripts L and R designate, respectively, sections just to the left and just to the right of the support. The amplitude of the exciting couple at support $\underline{z}$ is denoted by $\overline{M}_z$.

Fundamental Equations

Figure 2 shows, in an exaggerated scale, the extreme deflected position of spans j-1 and j of a continuous beam undergoing steady-state forced vibration. The vibration is assumed to be maintained by a single exciting couple applied at support $\underline{z}$. The rotations and bending moments at the ends of each span are indicated in their positive directions. Their value at a time $\underline{t}$ for support $\underline{j}$ are

$$\theta_j(t) = \theta_j \cos \omega t \quad \text{and} \quad M_j(t) = M_j \cos \omega t. \qquad (6)$$

In the equations to be used, the term $\cos \omega t$ appears as a common factor; for convenience this will be omitted, and in the remainder of this discussion, the terms "amplitude of rotation" and "rotation" and the terms "amplitude of moment" and "moment" will be used interchangeably.

Consider the interior support $\underline{j}$. From the conditions of continuity and equilibrium it follows that

$$(\theta_j)_L = (\theta_j)_R = \theta_j , \qquad (7)$$

and

$$(M_j)_L + (M_j)_R = 0. \qquad (8)$$

The moments $(M_j)_L$ and $(M_j)_R$ can now be expressed in terms of the end

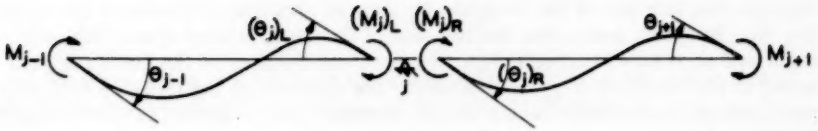
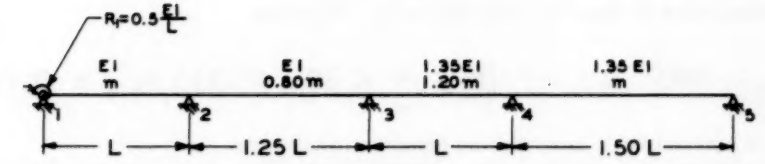
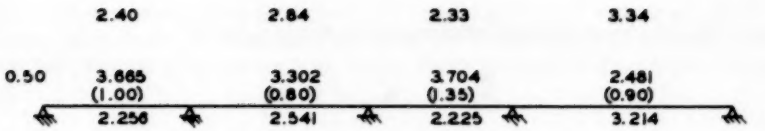


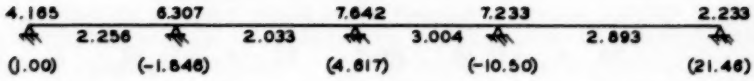
FIG. 2 SPANS j -1 AND j OF A CONTINUOUS BEAM



(a)



(b)



$$\bar{M}_5 = [2.233(21.46) + 2.893(-10.50)]\frac{EI}{L} = 17.54\frac{EI}{L}$$

(c)

FIG. 3 ILLUSTRATIVE EXAMPLE 1

rotations of spans j-1 and j, respectively. To start with, consider span j. Assume, first, that the right end of the span is held fixed while the left end is rotated through an angle θ_j; then, the moment at the end being rotated is equal to the product of the rotation θ_j and the flexural stiffness of the member K_j. Imagine, next, that the left end of the span is kept fixed while the right end is rotated through θ_{j+1}; the moment induced at the fixed left end is equal to the rotation θ_{j+1} multiplied by the product of the flexural stiffness and flexural carry-over factor of the member $(kK)_j$. Since the governing differential equation is linear and the principle of super-position is valid,

$$(M_j)_R = K_j\theta_j + (kK)_j\theta_{j+1}. \qquad (9a)$$

By considering span j-1, one obtains in a similar manner the result

$$(M_j)_L = K_{j-1}\theta_j + (kK)_{j-1}\theta_{j-1}. \qquad (9b)$$

Substitution of Eqs. (9a) and (9b) in Eq. (8) yields

$$(kK)_{j-1}\theta_{j-1} + [K_{j-1} + K_j]\theta_j + (kK)_j\theta_{j+1} = 0. \qquad (10a)$$

It is convenient to rewrite this equation in the form

$$\theta_{j+1} = -\frac{(kK)_{j-1}\theta_{j-1} + \bar{K}_j\theta_j}{(kK)_j}, \qquad (10b)$$

where $\bar{K}_j$, the total flexural stiffness at support j, is

$$\bar{K}_j = K_{j-1} + K_j. \qquad (11)$$

The foregoing expressions are, strictly speaking, applicable to interior supports only. For the end supports, the equations have the following specialized forms. Consider the general case in which the extreme ends of the beam are elastically restrained against rotation. The relationship between the restraining end moments and the corresponding rotations are

$$M_1 = -R_1\theta_1 \quad \text{and} \quad M_2 = -R_2\theta_2, \qquad (12)$$

where R_1 and R_2 are known stiffnesses of the rotational restraints at the left and the right ends, respectively. For a hinged end R = 0, and for a fixed end R = infinity. The negative signs in the foregoing expressions denote that for a positive restraint (positive value of R), the restraining moment acts in a direction opposite to the direction of rotation of the span.

Consider first the extreme left end of the beam. For a hinged or partially fixed condition, Eq. (10a) reduces to

$$\bar{K}_1\theta_1 + (kK)_1\theta_2 = 0, \qquad (13a)$$

where

$$\bar{K}_1 = R_1 + K_1 . \tag{13b}$$

For a fixed condition, $\theta_1 = 0$, and the relationship between the internal moment at the fixed end and the rotation at the second support is

$$M_1 = (kK)_1 \theta_2 . \tag{14}$$

Consider finally the right end of the beam. For a hinged or partially fixed end, the magnitude of the exciting couple is obtained from a specialized form of Eq. (10a) as

$$\bar{M}_z = (kK)_{z-1} \theta_{z-1} + \bar{K}_z \theta_z \tag{15a}$$

where

$$\bar{K}_z = K_{z-1} + R_z \tag{15b}$$

At a natural frequency $\bar{M}_z$ must be identically equal to zero. For a fixed end

$$\theta_z = 0 , \tag{16}$$

and the criterion for a natural frequency is that Eq. (16) be satisfied. The magnitude of the internal bending moment at the fixed end is of no practical interest but, should it be desired, it can be calculated from Eq. (15) by taking $\theta_z = 0$.

Outline of the Procedure

As previously remarked, the principal computational task of the procedure is the determination of the exciting force at the right end of the beam necessary to maintain a given amplitude of rotation (or internal bending moment) at the left end. Since the natural frequencies of a system depend on the relative rather than the absolute values of the displacements, any arbitrary amplitude, consistent with the actual boundary conditions may be chosen for the amplitude of vibration at support 1. For a hinged or a partially fixed end, θ_1 is taken, for convenience, equal to unity; for a fixed end, θ_1 is equal to zero, and M_1 or M_1 times the L/EI of some reference span $\underline{r}$ is taken equal to unity instead.

The procedure may now be outlined as follows:

1. Start by assuming a value of λ for some reference span $\underline{r}$; this step is equivalent to assuming a frequency of vibration

$$\omega = \frac{\lambda_r^2}{L_r^2} \sqrt{\frac{E_r I_r}{m_r}} .$$

Compute then the values of λ for the remaining spans from the expression

$$\lambda_j = \sqrt[4]{\frac{m_j}{m_r}\frac{E_r I_r}{E_j I_j}}\,\frac{L_j}{L_r}\,\lambda_r \qquad (17)$$

The pertinent calculations are conveniently carried out in a tabular form, as shown in the following illustrative example. Equation (17) is obtained readily by equating the expression for ω of the r-th span to that of the j-th span.

2. From the table in the Appendix, obtain the values of $\underline{K}$ and $\underline{kK}$ for the various spans.

3. Compute the rotation of the beam over the second support from Eq. (13) or (14) and, by successive applications of Eq. (10b), evaluate the rotations θ_3 to θ_z. A convenient scheme for arranging the computations is described in the illustrative example. If support $\underline{z}$ is fixed, the determination of the rotation θ_z completes one cycle of the procedure (see Eq. 16). However, if this support is hinged or only partially fixed, it is necessary to carry out the additional step of evaluating Eq. (15).

4. Repeat steps 1 through 3 for different values of λ_r, and plot the values calculated for θ_z or $\overline{M}_z$ as a function of λ_r. The zero intercepts of the resulting curve which, in general, has the shape shown in Fig. 4, corresponds to the natural frequencies of the system considered.

Illustrative Example

It is desired to calculate the first five natural frequencies of the four-span continuous beam shown in Fig. 3a. The beam is elastically restrained against rotation at the left end and simply supported at the right end. The stiffness of the end restraint and the properties of the various spans are shown in Fig. 3a.

For convenience in carrying out the computations, the natural frequencies of the system will be expressed in terms of the pertinent properties of a reference span $\underline{r}$. In this case, we take r = 1. In terms of the EI/L of the r-th span, the values of $\underline{K}$ and $\underline{kK}$ for a span $\underline{j}$ are equal to those obtained from the Appendix multiplied by the dimensionless factor

$$\alpha_j = \frac{E_j I_j}{E_r I_r}\frac{L_r}{L_j} \,. \qquad (18)$$

The quantities λ_j/λ_r and α_j are evaluated in Table 1. It should be noted that these calculations are independent of the frequency of vibration. A complete cycle of the procedure will now be presented for a trial value of $\lambda_r = \lambda_1 = 2.40$; this value corresponds to a circular frequency of vibration

$$\omega = \frac{(2.40)^2}{L^2}\sqrt{\frac{EI}{m}} \,.$$

The values of λ for the various spans are recorded at the top of the diagram in Fig. 3b. These values are obtained as the product of the assumed λ_r and each of the values in column 5 of Table 1. The values of α for the

TABLE 1

Span	(1) $\frac{m_j}{m_r}$	(2) $\frac{E_jI_j}{E_rI_r}$	(3) $\sqrt[4]{\frac{(1)}{(2)}}$	(4) $\frac{L_j}{L_r}$	(5) $\frac{\lambda_j}{\lambda_r} = (3)(4)$	(6) $\alpha_j = \frac{(2)}{(4)}$
1 = r	1.00	1.00	1.00	1.00	1.00	1.00
2	0.80	1.00	0.9457	1.25	1.182	0.80
3	1.20	1.35	0.9710	1.00	0.9710	1.35
4	1.00	1.35	0.9277	1.50	1.392	0.90

various members are recorded in parentheses on the same diagram. The numbers above the values of α give the stiffnesses of the various spans, whereas those below the values of α give the corresponding products of the stiffnesses and the carry-over factors. These quantities are expressed in terms of the EI/L of the span to which they refer, and they are obtained directly from the Appendix.

Before using them in the fundamental equations, it is convenient to express the quantities $\underline{K}$ and $\underline{kK}$ in terms of a common factor. On the diagram in Fig. 3c, these quantities are given in terms of the EI/L of the reference span. On this diagram, the number above each support represents the value of $\overline{K}$ for that support, and the number at the middle of each span represents the value of $\underline{kK}$ for that span. The value of $\overline{K}$ for the j-th support is determined by taking the sum of the products of the α values and the $\underline{K}$ values in Fig. 3b for spans j-1 and j. Thus, the total stiffness at support 3 is

$$\overline{K}_3 = \left[3.302(0.80) + 3.704(1.35)\right]\frac{EI}{L} = 7.642\,\frac{EI}{L}\ .$$

Similarly, the values of $\underline{kK}$ are obtained by multiplying the values of $\underline{kK}$ given in Fig. 3b by the corresponding values of α.

With the foregoing information established, one can compute readily the rotations of the beam. On the diagram in Fig. 3c, the number in parentheses below each support represents the value of the rotation at that support. The rotation θ_1 is taken equal to unity. The rotation θ_2 is evaluated from Eq. (13a) which, it should be recalled, is a specialized form of Eq. (10b). Thus,

$$\theta_2 = -\frac{4.165(1.00)}{2.236} = -1.846.$$

The rotations θ_3, θ_4 and θ_5 are determined by the application of Eq. (10b) to joints 2, 3 and 4, respectively. Thus,

$$\theta_3 = -\frac{2.256(1.00) + 6.307(-1.846)}{2.033} = 4.617,$$

$$\theta_4 = -\frac{2.033(-1.846) + 7.642(4.617)}{3.004} = -10.50,$$

$$\theta_5 = -\frac{3.004(4.617) + 7.233(-10.50)}{2.893} = 21.46.$$

These results are recorded on the diagram as they are computed.

The exciting couple, evaluated at the bottom of the diagram in Fig. 3c by use of Eqs. (15), is found to be equal to 17.54 EI/L.

Since $\overline{M}_5$ is different from zero, the assumed value of $\lambda_1 = 2.40$ does not correspond to a natural frequency of the system. The physical significance of the computed value of 17.54 EI/L is as follows: the negative of this value divided by the rotation θ_5,

$$-\frac{17.54}{21.46}\frac{EI}{L} = -0.8173\frac{EI}{L},$$

represents the stiffness of a rotational constraint which, if applied at the right end of the beam, would make the assumed frequency correspond to a natural frequency of the system.

By repeating several such cycles of computation for different values of λ_1, the curve shown in Fig. 4 was obtained. The zero intercepts of this curve correspond to the desired natural frequencies. The first five critical values of λ are recorded in the figure. The corresponding circular natural frequencies ω_N, in radians per second, are

$$(\omega_N)_1 = \frac{6.27}{L^2}\sqrt{\frac{EI}{m}}, \qquad (\omega_N)_2 = \frac{9.42}{L^2}\sqrt{\frac{EI}{m}},$$

$$(\omega_N)_3 = \frac{13.7}{L^2}\sqrt{\frac{EI}{m}}, \qquad (\omega_N)_4 = \frac{16.9}{L^2}\sqrt{\frac{EI}{m}}.$$

$$(\omega_N)_5 = \frac{24.0}{L^2}\sqrt{\frac{EI}{m}}.$$

The natural frequencies, in cycles per second, may be computed from the relation

$$f_N = \frac{\omega_N}{2\pi}. \qquad (19)$$

In this example, and in the examples to follow, the calculations have been carried out to four significant figures. In the particular cycle described, this accuracy was greater than actually required. In general, for the determination of the lower natural frequencies of continuous beams slide-rule accuracy will prove satisfactory. However, for the higher natural frequencies, particularly in the case of frames involving closed panels, the computations may involve small differences between large quantities, and it may be necessary to retain a minimum of four significant figures.

Part III. Continuous Frames Without Sidesway

The following discussion applies to rigid jointed, plane, elastic frameworks for which the joints do not translate. The individual members of each frame are considered to be of uniform cross section and mass per unit of length.

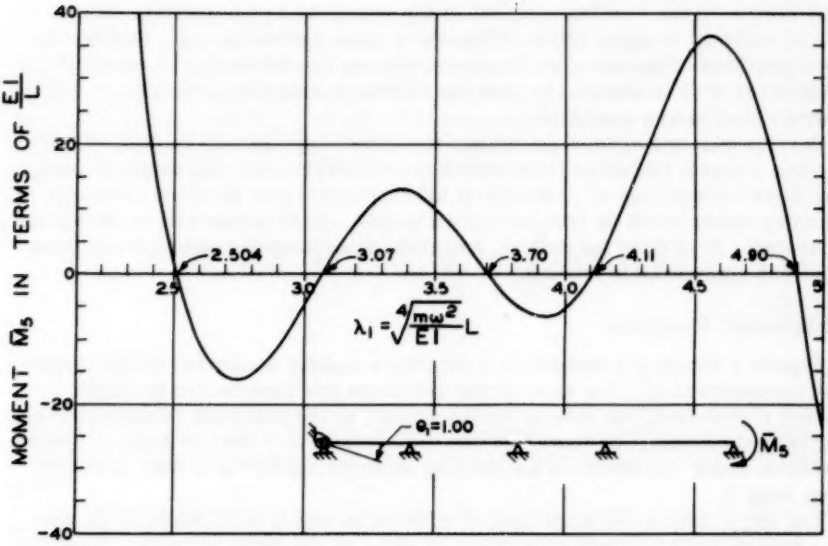


FIG. 4 VARIATION OF EXCITING MOMENT AS A FUNCTION OF λ_1, EXAMPLE 1

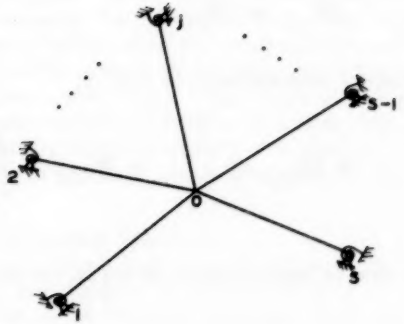


FIG. 5 TYPICAL JOINT OF A FRAME

The vibrations are assumed to take place in the plane of the framework. The change in length of the members due to axial deformation and the effect of the axial forces on the bending moment in the members are neglected. In addition, no account is taken of the influence of axial vibrations. As before, the cross sectional dimensions of the members are considered to be small in comparison to their length, so that the effects of shearing deformation and rotatory inertia are negligible.

For the purpose of this discussion structural frames will be divided into: (a) open frames, including those which do not involve any closed panels and have known conditions of restraint at all terminals, and (b) closed frames, including those which do involve closed panels. Each group will be discussed separately. It is first necessary, however, to develop the appropriate slope equations applicable to frames.

Fundamental Equations

Figure 5 shows s members of a structure rigidly connected at their common intersection 0. The ends of the members are assumed to be fixed against translation, but free to rotate subject to the restraint imposed by the adjoining members. Assume that the structure is in a steady-state of forced vibration under the action of an exciting moment applied at a joint different from joint 0.

Let $\theta_{0,j}$ denote the amplitude of rotation at end 0 of member (0,j), and $\theta_{j,0}$ denote the amplitude of rotation at end j of the same member. Similarly, let $M_{0,j}$ and $M_{j,0}$ be the corresponding moment amplitudes at the same ends. Also, let $K_{0,j}$ and $(kK)_{0,j}$ be, respectively, the values of K and kK for member (0,j).

Since all members are rigidly connected at their joints

$$\theta_{0,1} = \theta_{0,2} = \ldots = \theta_{0,j} = \ldots = \theta_{0,s} = \theta_0 , \quad (20a)$$

and

$$\theta_{j,0} = \theta_j . \quad (20b)$$

Furthermore, since no external moment acts at joint 0,

$$M_{0,1} + M_{0,2} + \ldots + M_{0,j} + \ldots + M_{0,s} = \sum_{j=1}^{s} M_{0,j} = 0. \quad (21)$$

The moment $M_{0,j}$ may now be expressed in terms of the end rotations of member (0,j) as follows:

$$M_{0,j} = K_{0,j}\theta_0 + (kK)_{0,j}\theta_j . \quad (22)$$

Substituting this expression into Eq. (21), one obtains Eq. (23a)

$$\sum_{j=1}^{s} K_{0,j}\theta_0 + \sum_{j=1}^{s} (kK)_{0,j}\theta_j = 0, \quad (23a)$$

which expresses the conditions of both equilibrium and continuity for joint 0 of the structure. If only two members meet at joint 0, Eq. (23a) reduces to Eq. (10a) for continuous beams.

If the restraints at the far ends of the members meeting at a joint are known, it is convenient to use effective stiffnesses. Assume that the restraints at ends 1 and 2 of the portion of the structure shown in Fig. 5 are known. Let $K'_{0,1}$ and $K'_{0,2}$ be the effective stiffnesses of members (0,1) and (0,2), respectively; these may be determined from Eq. (3). Then, Eq. (23a) may be written as

$$(K'_{0,1} + K'_{0,2})\theta_0 + \sum_{j=3}^{k} K_{0,j}\,\theta_0 + \sum_{j=3}^{k} (kK)_{0,j}\,\theta_j = 0 \qquad (23b)$$

Open Frames

Simple L-frames and portal frames, such as those shown in Figs. 6a and 6b, behave as continuous beams on rigid supports. Therefore, their natural frequencies can be calculated by the procedure presented in Part II of this paper.

The natural frequencies of open frames, such as those shown in Figs. 6c through 6e, can be evaluated by a variation of the procedure applicable to beams. Consider any of the frames in Figs. 6c through 6e. Let 1 denote the joint of the frame at the extreme left terminal and $\underline{z}$ denote the joint at the extreme right terminal. Without loss of generality, it may be assumed that joint $\underline{z}$ is either hinged or elastically restrained. A fixed end may be treated in the same manner as a fixed end of a continuous beam. Assume that the structure is in a steady-state of forced vibration under the action of an exciting couple $\overline{M}_{\underline{z}}$ applied at joint $\underline{z}$. The amplitude of the slope or of the bending moment at joint 1 is assumed to have a fixed value. If the joint is hinged or is elastically restrained, the amplitude of the slope is taken equal to unity. If the joint is fixed, the amplitude of the internal bending moment is taken equal to unity instead. For an assumed frequency of vibration, it is generally possible to calculate the magnitude of the exciting moment $\overline{M}_{\underline{z}}$ in a manner entirely analogous to that used for continuous beams, by working progressively from one end of the frame to the other, with the only difference that one must use the more general Eqs. (23) rather than Eqs. (10). By repeating this procedure for several frequencies of vibration, the magnitude of $\overline{M}_{\underline{z}}$ may be plotted as a function of the frequency of vibration. The frequencies for which the amplitude of the exciting couple vanishes are the natural frequencies of the frame.

Natural Modes of Partial Vibration

When a continuous beam is in a natural mode of vibration, each span oscillates with finite amplitudes. In the case of frames, however, it is possible for only a portion of the frame to oscillate while the rest remains stationary (vibrates with zero amplitude). Figure 7 presents several such modes for the structures shown in Figs. 6c through 6e. Such modes can, of course, exist only if the dimensions of the various members satisfy certain definite relations. Natural modes for which only a portion of a structure vibrates while the rest remains still will be referred to as "natural modes of partial vibration."

Because of the assumptions on which it is based, the procedure described

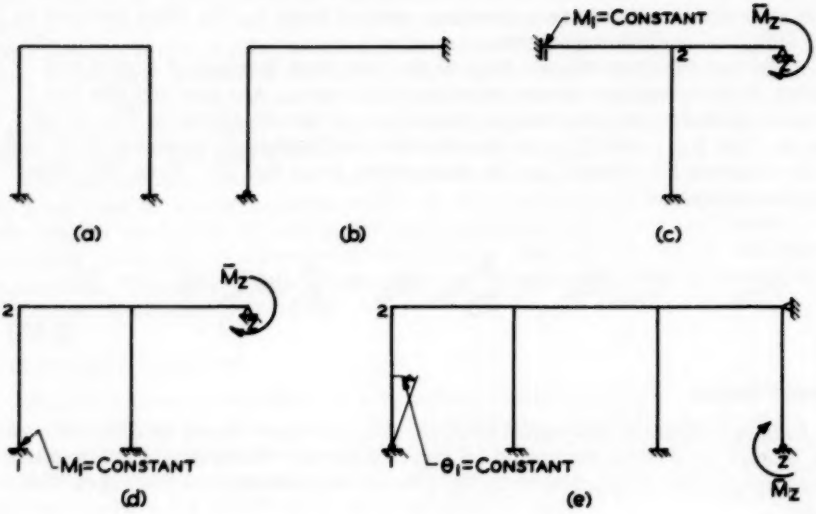


FIG. 6 TYPICAL OPEN FRAMES

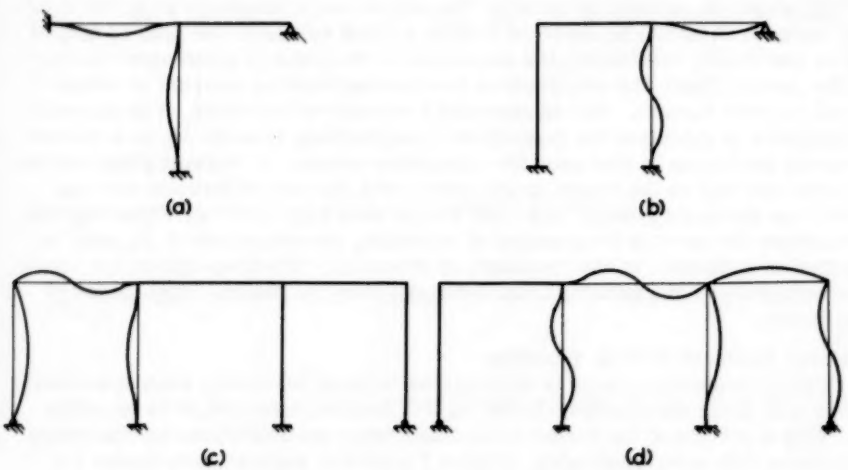


FIG. 7 POSSIBLE NATURAL MODES OF PARTIAL VIBRATION

thus far may fail to detect natural frequencies corresponding to natural modes of partial vibration. It may be recalled that the amplitude of the slope or bending moment at joint 1 of the structure is assumed to be fixed in the procedure. Accordingly, the procedure will fail to reveal those natural frequencies for which bar (1,2) remains stationary. A second assumption implicit in the procedure is that the rotation of joint z is different from zero. Since this condition is not always satisfied, the procedure will fail also to reveal the natural frequencies for which the bars meeting at joint z are stationary.

The natural frequencies which the main procedure fails to detect may be determined by (a) calculating the frequencies for which bar (1,2) and the bars meeting at joint z can remain still, and (b) ascertaining whether or not these frequencies are natural frequencies of the system. The details of this supplementary procedure are described by a numerical example in the following section.

It is worthy of note that natural modes of partial vibration occur rather infrequently; they are most likely to occur in the case of frames involving panels of uniform members. It is worthy of note also that for most practical applications, it may be entirely unnecessary to calculate natural frequencies corresponding to such modes. In practice, one is usually interested in determining natural frequencies comprised within specified ranges of frequencies. Therefore, if the values calculated in the first step of the supplementary procedure are found to lie outside the ranges of interest, it will be superfluous to carry out the second step. The first step of this procedure, as will be seen shortly, can usually be carried out almost by inspection. It should finally be noted that natural modes of partial vibration correspond always to higher natural frequencies. Therefore, no consideration need be given to these modes, if only the fundamental natural frequency of a frame is to be determined.

Illustrative Example

As an illustration, we shall calculate the first few natural frequencies of the simple frame shown in Fig. 8a. The bars identified by (1) are taken identical, while the bar designated by (2) is considered to have such dimensions that

$$E_2 I_2 / L_2 = E_1 I_1 / L_1, \quad L_2 = 0.80\, L_1,$$

and

$$\lambda_2 = \frac{3.927}{4.730} \qquad \lambda_2 = 0.8302\, \lambda_1 .$$

The subscripts 1 and 2 refer to bars (1) and (2), respectively. These proportions have been chosen so that the frame can vibrate in a natural mode of partial vibration.

For the sake of brevity, only one cycle of the procedure is presented. The computations are given for a value of $\lambda_1 = 3.30$, which corresponds to a value of $\lambda_2 = 2.74$.

The appropriate values of K and kK, obtained from the Appendix, are:

for bars (1), $K_1 = 2.572\, E_1 I_1 / L_1$ and $(kK)_1 = 3.138\, E_1 I_1 / L_1$;

for bar (2), $K_2 = 3.405\, E_1 I_1 / L_1$ and $(kK)_2 = 2.459\, E_1 I_1 / L_1$.

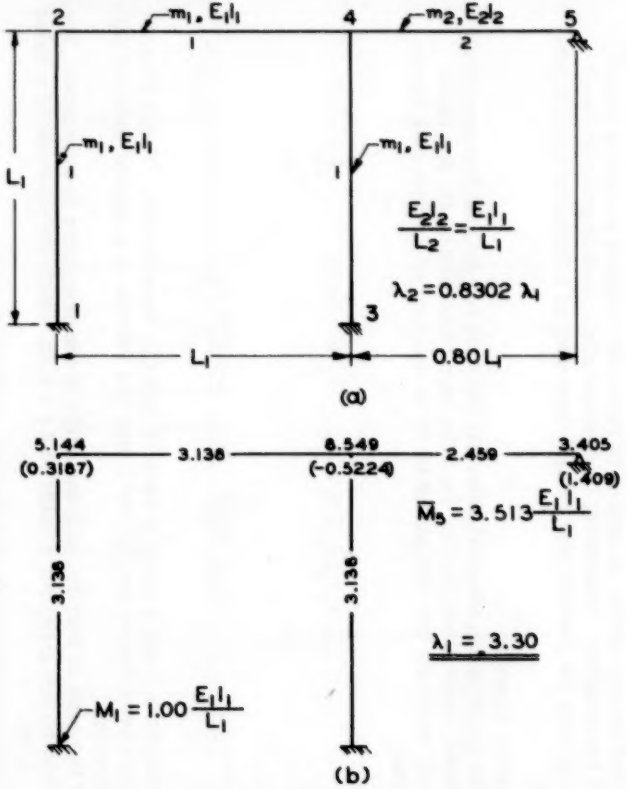


FIG. 8 ILLUSTRATIVE EXAMPLE 2

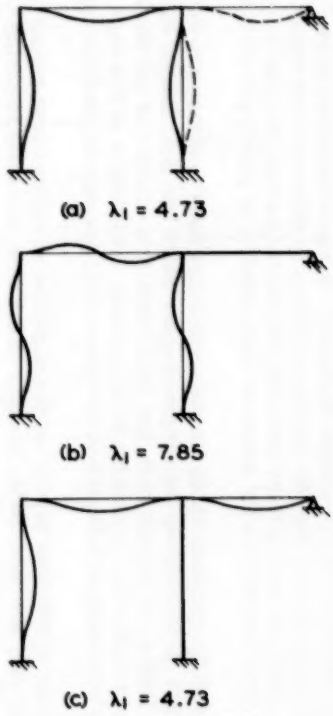


FIG. 10 NATURAL MODES OF PARTIAL VIBRATION FOR EXAMPLE 2

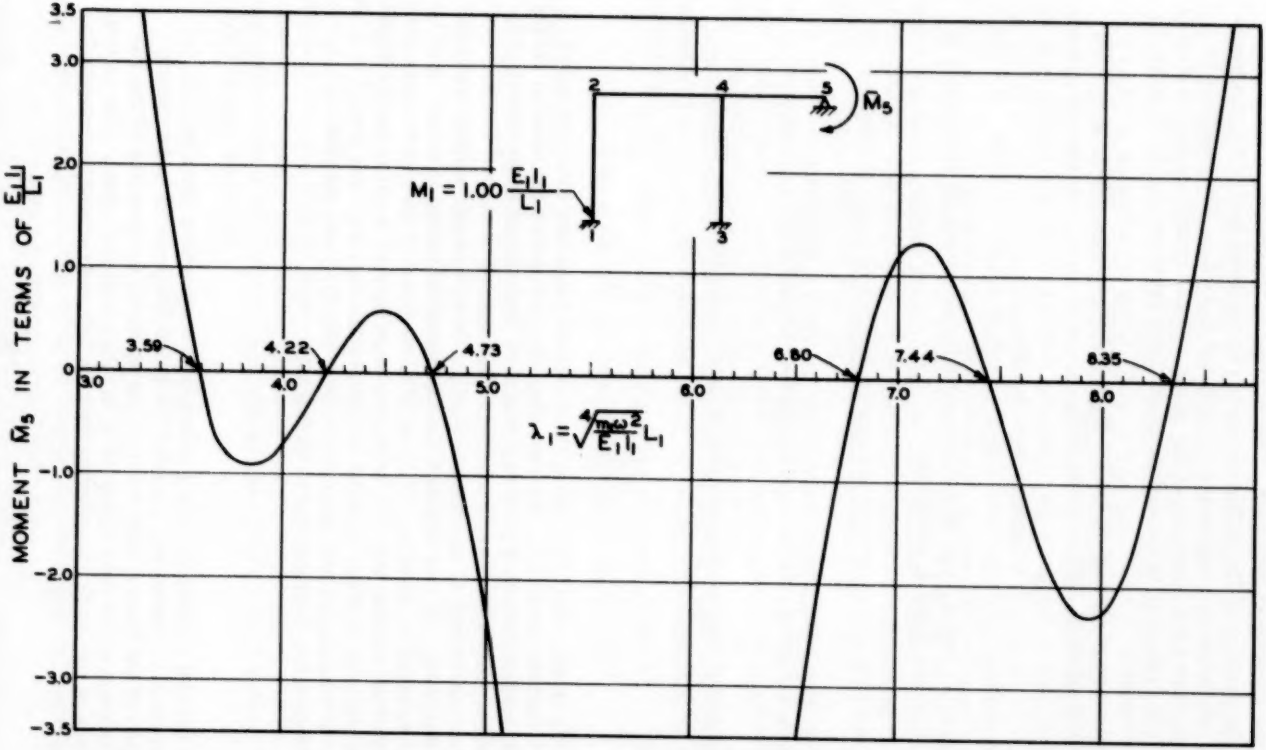


FIG. 9 VARIATION OF EXCITING MOMENT AS A FUNCTION OF λ_1, EXAMPLE 2

The data necessary in the analysis are compiled in Fig. 8b. The number without parentheses opposite each joint gives the sum of the stiffnesses of the members meeting at that joint. The number at the middle of each member gives the product of the stiffness and the carry-over factor for the member. Both quantities are expressed in terms of E_1I_1/L_1. The numbers in parentheses denote the rotations of the joints. These quantities are evaluated in the manner described in the following, and are recorded on the diagram as they are computed.

The procedure is started by taking $M_1 = 1.00\ E_1I_1/L_1$. Then θ_2 is computed by application to joint 1 of Eq. (22),

$$\theta_2 = \frac{1.00}{3.138} = 0.3187 .$$

The rotations θ_4 and θ_5 are then determined successively by application of Eq. (23a) to joints 2 and 4, as follows

$$\theta_4 = -\frac{3.138\ (0) + 5.144\ (0.3187)}{3.138} = -0.5224,$$

$$\theta_5 = -\frac{3.138\ (0.3187) + 3.138\ (0) + 8.549\ (-0.5224)}{2.459} = 1.409.$$

The magnitude of the exciting moment is determined from Eq. (22) as

$$\overline{M}_5 = \left[3.405\ (1.409) + 2.459\ (-0.5224)\right]\frac{E_1I_1}{L_1} = 3.513\ \frac{E_1I_1}{L_1} .$$

Since $\overline{M}_5$ is different from zero, the assumed frequency does not correspond to a natural frequency. By repeating this procedure for several values of λ_1, the curve shown in Fig. 9 was obtained. The values of λ_1 corresponding to natural frequencies are recorded on the figure.

It is next necessary to investigate whether there are any natural frequencies for which $M_1 = 0$. To this end, the following reasoning is used. In order for M_1 to be equal to zero, bar (1,2) must be stationary; then, the rotation and the internal bending moment at joint 2 must be also equal to zero. This condition requires that bar (2,4) be stationary and that $\theta_4 = M_{4,2} = 0$. But with joint 4 remaining fixed against rotation, bar (3,4) can oscillate freely only at frequencies represented by values of

$$\lambda_1 = 4.730,\ 7.853,\ \ldots \qquad (24)$$

During a natural frequency, every member of a structure vibrates with the same frequency. Therefore, in order for these frequencies to be natural frequencies of the frame, they must be also natural frequencies of the remaining portion of the frame (bar (4,5) in this particular case). The natural frequencies of bar (4,5), considering that its left end is fixed, correspond to values of

$$\lambda_2 = 3.927,\ 7.069,\ \ldots \qquad (25a)$$

which, in turn, correspond to values of

$$\lambda_1 = 4.730,\ 8.514,\ \ldots \qquad (25b)$$

Comparing the values given in Eq. (25b) with those given in Eq. (24) one concludes that, within the range of frequencies considered in Fig. 9, the values $\lambda_1 = 7.853$ and $\lambda_1 = 8.514$ do not correspond to natural frequencies, whereas the value $\lambda_1 = 4.730$ does. The natural mode of partial vibration for $\lambda_1 = 4.730$ is shown by the solid curve in Fig. 10a.

Finally, it is necessary to investigate whether there are any natural frequencies for which $\theta_5 = 0$. One reasons that θ_5 can be equal to zero only if bar (4,5) is stationary. Under this condition, $\theta_4 = M_{4,5} = 0$; then, bar (3,4) can vibrate freely only at frequencies represented by the values of λ_1 given in Eq. (24). Since members (1,2) and (2,4) of the remaining portion of the frame are identical to member (3,4), each of the members composing the panel (1,2,3,4) can vibrate in a fixed-ended condition for the same frequency; therefore, the λ_1 values given in Eq. (24) correspond to natural frequencies of the frame. The natural mode of partial vibration corresponding to $\lambda_1 = 4.730$ is shown by the dotted curve in Fig. 10a, and the mode corresponding to $\lambda_1 = 7.853$ is shown in Fig. 10c.

It should be noted that a total of three natural modes have been determined for a frequency corresponding to a value of $\lambda_1 = 4.730$. It can be shown, however, that of these three modes only two are independent, the mode shown in Fig. 10b being a linear combination of the other two.

From the foregoing, one concludes that, within the range of frequencies considered in Fig. 9, the complete set of λ_1 values corresponding to natural frequencies is

3.59, 4.22, 4.73 (double), 6.80, 7.44, 7.85, and 8.35

Closed Frames

The fundamental features of the procedure for calculating natural frequencies of closed frames are described in this section by reference to the two-celled rectangular frame shown in Fig. 11a.

The procedure is started by assuming that the frame is cut at a convenient joint. For the example considered, the cut is introduced at joint z = 6, as shown in the figure. It is next assumed that the frame undergoes a steady-state of forced vibration with finite amplitudes and known frequency under the action of an exciting couple applied at the cut joint (z = 6). In general, there will be a discontinuity in slope in the cut joint. However, for a natural frequency of vibration, both the amplitude of the exciting moment and the discontinuity in slope will vanish. Therefore, the criterion for a natural frequency is that equations

$$\theta_{6,4} - \theta_{6,5} = 0$$
$$\bar{M}_6 = 0 \qquad (26a)$$

be satisfied simultaneously. It is, of course, assumed that the frame does not oscillate in a natural mode of partial vibration.

Determination of the Discontinuity in Slope and of the Exciting Moment

The discontinuity in slope and the magnitude of the exciting moment may be determined by working progressively from joint to joint across the structure, in the same general way as for open frames, except for one major

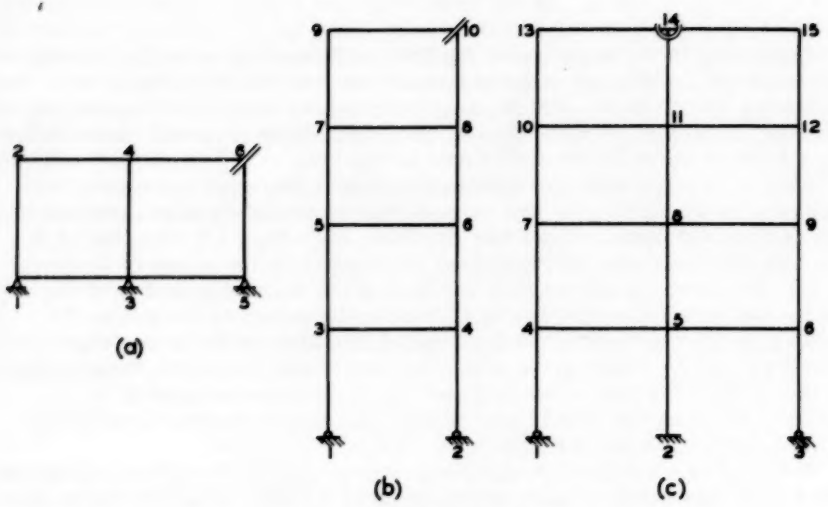


FIG. 11 TYPICAL CLOSED FRAMES, NO SIDESWAY

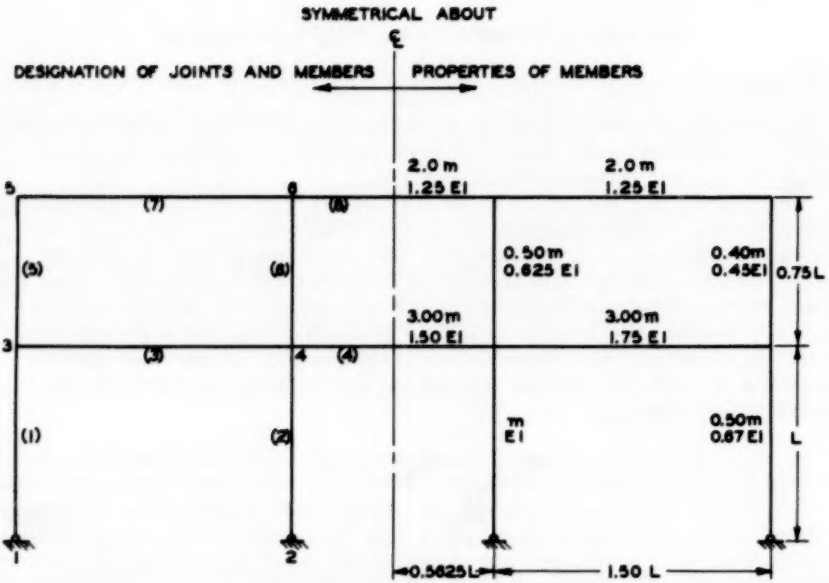


FIG. 12 ILLUSTRATIVE EXAMPLE 3

difference. Whereas for the cases considered thus far the equation for the first joint involved only two unknowns, in the present case, the corresponding equation involves a minimum of three unknowns.

To illustrate this point, reference is again made to the frame shown in Fig. 11a. One starts by applying Eq. (23a) to joint 1. Then, the resulting expression will involve the three unknowns, θ_1, θ_2, and θ_3, only one of which can be taken equal to a constant. Thus, it becomes necessary to express θ_3 in terms of both θ_1 and θ_2. Next, one selects consecutive joints in such an order that each new equation involves only one additional unknown, which is also expressed in terms of both θ_1 and θ_2. The rotation θ_4 is determined by applying Eq. (23a) to joint 2. The rotations θ_5, $\theta_{6,4}$ and $\theta_{6,5}$ are determined by the successive application of the same equation to joints 3, 4 and 5, respectively. Finally, the exciting moment $\overline{M}_z$ (z = 6) is obtained from Eq. (22). Having expressed all these quantities in terms of θ_1 and θ_2, one may write Eq. (26a) in the form

$$c_{11}\theta_1 + c_{12}\theta_2 = 0,$$
$$c_{21}\theta_1 + c_{22}\theta_2 = 0. \qquad (26b)$$

In these expressions, the c's are numerical constants, the magnitudes of which depend on the assumed frequency of vibration and on the properties of all the members of the structure. Equations (26b) are a set of linear, homogeneous equations for the unknowns θ_1 and θ_2, and have a solution different from zero when the determinant of the coefficients of θ_1 and θ_2 is zero:

$$\Delta_6(\theta_1, \theta_2) = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = 0. \qquad (27)$$

It should be noted that, in this procedure, the cut is always introduced at the joint, the rotation of which is evaluated last by the scheme described. Furthermore, the number of members meeting at the cut are always equal to the number of unknowns necessary to express the rotations of the joints.

The foregoing criterion of the vanishing determinant will fail to reveal the natural frequencies for which θ_1 and θ_2 are simultaneously equal to zero. It will fail to reveal also the natural frequencies for which the bars meeting at the cut are stationary. If necessary, the natural frequencies corresponding to these conditions of partial vibration may be calculated by the supplementary procedure described previously.

Consider now the frames shown in Figs. 11b and 11c. The natural frequencies of vertical vibration of the single-bay, multi-story building frame shown in Fig. 11b may be found in identically the same manner as that described for the frame in Fig. 11a. The rotations may be expressed in terms of θ_1 and θ_2, and the cut may be introduced at joint 9 or 10. For the two-bay, multi-story frame shown in Fig. 11c, one needs to express the rotations of the joints in terms of three quantities, say θ_1, M_2 and θ_3, so that succeeding equations involve only one additional unknown. The cut is introduced at joint 14. The conditions of continuity and equilibrium at the cut may then be expressed as

$$\theta_{14,13} - \theta_{14,11} = c_{11}\theta_1 + c_{12}M_2 + c_{13}\theta_3 = 0$$
$$\theta_{14,11} - \theta_{14,15} = c_{21}\theta_1 + c_{22}M_2 + c_{23}\theta_3 = 0 \qquad (28)$$
$$\bar{M}_{14} = c_{31}\theta_1 + c_{32}M_2 + c_{33}\theta_3 = 0$$

where the c's are again numerical constants. Therefore, the criterion for a natural frequency is

$$\Delta_{14}(\theta_1, M_2, \theta_3) = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = 0. \qquad (29)$$

It is seen that, for this problem, it becomes necessary to evaluate a third order determinant. In general, the order of the determinant to be evaluated is equal to the number of unknowns used to determine the rotations of the joints.

Simplified Approach

Even if only two unknowns are needed to express the rotations of the joints, the computational work required to calculate the exciting moment and the discontinuity of slope at the cut may get fairly involved. One may remarkably simplify this work by determining the desired quantities as a linear combination of partial solutions. For illustration, refer again to the frame shown in Fig. 11a. First, assume that $\theta_1 = 1.00$ and that $\theta_2 = 0$. If now Eq. (23a) is applied to joint 1, it will be found that the rotation of joint 3 will be the only unknown in the resulting expression. Determine this unknown and, proceeding from joint to joint across the frame in the order described previously, evaluate the rotations of the remaining joints and designate them by θ'. Calculate also the magnitude of the exciting moment corresponding to the rotations θ' and designate it by $\bar{M}'_z$. Next, assume that $\theta_1 = 0$ and that $\theta_2 = 1.00$ and, in the same manner, calculate the rotations of the joints and the corresponding exciting moment. Denote these quantities by θ'' and $\bar{M}''_z$, respectively. Then, the actual rotation at a joint j is

$$\theta_j = \theta'_j\theta_1 + \theta''_j\theta_2 . \qquad (30)$$

For the frame shown in Fig. 11a, the total discontinuity of slope at the cut is

$$\theta_{6,4} - \theta_{6,5} = (\theta'_{6,4} - \theta'_{6,5})\theta_1 + (\theta''_{6,4} - \theta''_{6,5})\theta_2 = 0, \qquad (31)$$

and the total exciting moment is

$$\bar{M}_6 = M'_6\theta_1 + \bar{M}''_6\theta_2 . \qquad (32)$$

The coefficients of θ_1 and θ_2 in these expressions correspond to the coefficients c in Eq. (26b). The direct combination of the foregoing partial effects is justified by the fact that the differential equation for steady-state forced vibration is linear.

Outline of Procedure

The details of the main procedure can now be summarized as follows. Some of the steps already described are repeated here for the sake of completeness.

1) Assume that the frame is cut at a suitable joint. Then for some member of the frame, say member r, assume a value of λ_r (See Eq. 5). This is equivalent to assuming a frequency of vibration ω.

2) From Eq. (17) compute the values of λ for the remaining members of the frame, and from the table in the Appendix, obtain the appropriate values of K and kk for each member.

3) The values of K and kk, as given in the Appendix, are expressed in terms of E_jI_j/L_j, where the subscript j refers to the particular member considered. Next express these quantities in terms of the EI/L of the reference member r, by multiplying them by the dimensionless factor α_j (See Eq. 18).

4) Compute the sum of the stiffnesses of the members meeting at the various joints and record them on a diagram of the frame. On the same diagram, record also the product of the stiffness and the carry-over factor for each member. These quantities must be expressed in terms of the EI/L of the same reference member r. A convenient scheme for arranging the computations is shown in the illustrative example presented in the next section.

5) Choose the unknowns in terms of which the distortions of the frame will be expressed. In general, the number of unknowns that must be selected is equal to the number of the main longitudinal members in the frame. Consider the first of these unknowns equal to unity and the other equal to zero. By working across the structure, in the manner described previously, compute the rotations of the joints. Denote these by θ'. Compute also the exciting moment at the cut and denote it by $\overline{M}_z'$.

6) Repeat step 5, taking the second unknown equal to unity and the others equal to zero. Denote the resulting rotations by θ'' and the corresponding exciting moment by $\overline{M}_z''$. In general, steps 5 and 6 must be repeated as many times as there are unknowns in the expressions for the distortions of the system.

7) Determine the total discontinuity of slope and the total exciting moment at the cut, and evaluate the determinant of the coefficients of the unknowns.

8) Repeat steps 1 through 7 for a number of values of λ_r and plot the variation of the determinant evaluated in step 7 as a function of λ_r. The values of λ_r for which the determinant becomes equal to zero correspond to the desired natural frequencies of the frame.

Illustrative Example

As an illustration of the application of the method of continuous closed frames, we shall calculate the first eight natural frequencies of symmetrical vibration of the symmetrical frame shown in Fig. 12. All columns are considered to be hinged at the base.

Since the frame is symmetrical, for symmetrical natural modes of vibration the joints of the frame, even though free from external restraining

forces, will not translate during free vibration. Accordingly, the procedure described is applicable to this problem. In the following, only one half of the frame will be considered. The effect of symmetry will be taken into account by using for the girders of the central bay modified stiffnesses K^s instead of the usual stiffnesses K.

The dimensions of the frame and some additional data pertinent in the analysis are assembled in Table 2. Member (2) of the frame is taken as the

TABLE 2

Member	(1) $\frac{m_j}{m_r}$	(2) $\frac{E_j I_j}{E_r I_r}$	(3) $\sqrt[4]{\frac{(1)}{(2)}}$	(4) $\frac{L_j}{L_r}$	(5) $\frac{\lambda_j}{\lambda_r} = (3)(4)$	(6) $\alpha_j = \frac{(2)}{(4)}$	(7) λ_j
1	0.50	0.67	0.9294	1.00	0.9294	0.6700	2.04
2 = r	1.00	1.00	1.000	1.00	1.000	1.000	2.20
3	3.00	1.75	1.144	1.50	1.716	1.167	3.78
4	3.00	1.50	1.189	1.125	1.338	1.333	2.94
5	0.40	0.45	0.9710	0.75	0.7282	0.6000	1.60
6	0.50	0.625	0.9457	0.75	0.7093	0.8333	1.56
7	2.00	1.25	1.125	1.50	1.687	0.8333	3.71
8	2.00	1.25	1.125	1.125	1.265	1.111	2.78

reference member. Columns (5) and (6) of Table 2 give, respectively, the ratio of λ_j/λ_2 (Eq. 17) and the values of α_j (Eq. 18) for each member. For the sake of brevity, only one cycle of the procedure is presented. Detailed computations are given for a value of $\lambda_2 = 2.20$, which corresponds to a circular frequency of vibration $\omega = \frac{(2.20)^2}{L^2}\sqrt{\frac{EI}{m}}$.

The values of α for the various members are recorded in parentheses on the diagram in Fig. 13a. The numbers above these values give the stiffnesses of the various members, while those below the values of α give the products of the stiffnesses and the carry-over factors. These quantities are expressed in terms of the EI/L of the member to which they refer, and they are obtained directly from the Appendix.

On the diagram in Fig. 13b, the number without parentheses opposite each joint represents the total stiffness of the members framing into that joint. These stiffnesses are expressed in terms of the EI/L of the reference member (2). Thus, the total stiffness of the members connecting at joint 4 is

$$\bar{K}_4 = \left[3.768(1.00) + 0.9188(1.167) + 3.943(0.8333) + 0.5451(1.333)\right]\frac{EI}{L} .$$

$$\bar{K}_4 = 8.853\,\frac{EI}{L} .$$

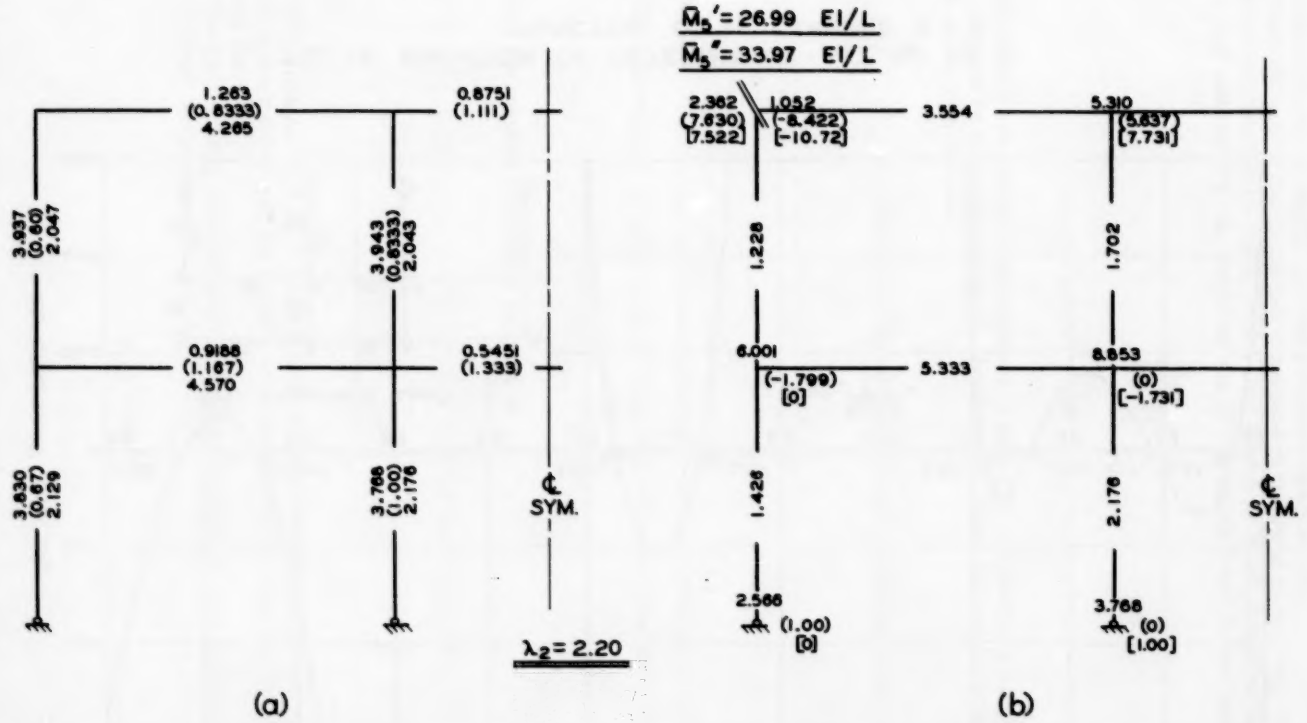


FIG. 13 ONE CYCLE OF PROCEDURE FOR EXAMPLE 3

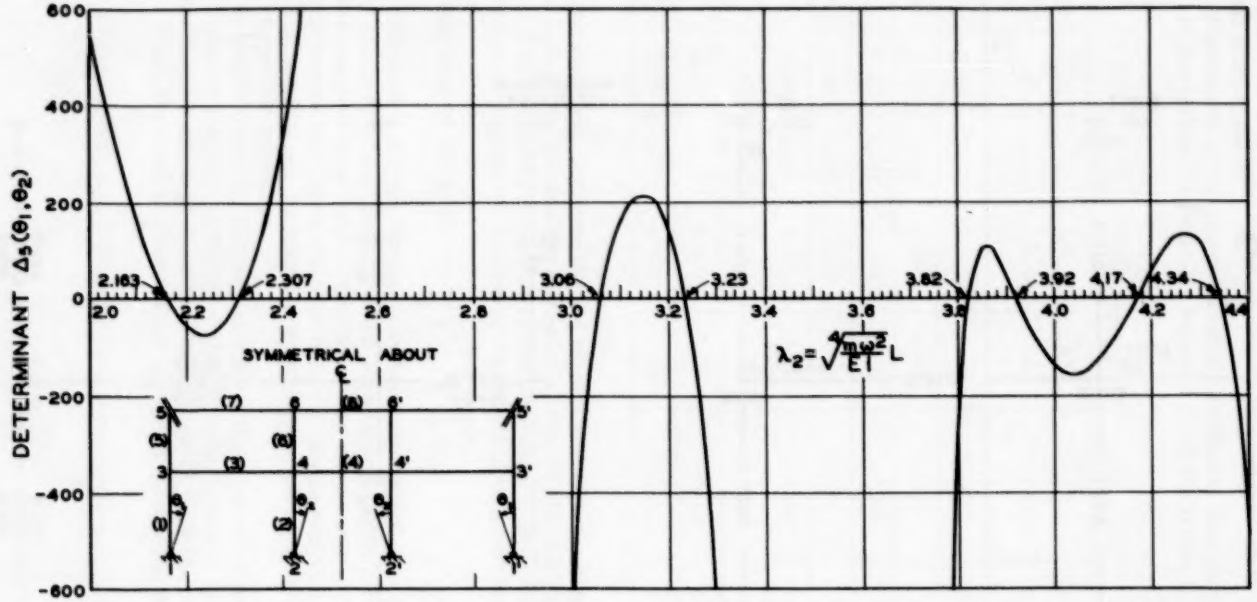


FIG. 14 VARIATION OF DETERMINANT $\Delta_5(\theta_1,\theta_2)$ AS A FUNCTION OF λ_2, EXAMPLE 3

The number at the middle of each member represents the product of the stiffness and the carry-over factor for that member; these values are also expressed in terms of the EI/L of the reference member (2).

The rotations of the joints are expressed in terms of θ_1 and θ_2. The frame is assumed to be cut at joint 5 (and joint 5'). First, consider that $\theta_1 = 1.00$ and that $\theta_2 = 0$. Applying Eq. (22) to joints 1 and 2, one finds that

$$\theta'_3 = -\frac{2.566(1.00)}{1.426} = -1.799 ,$$

$$\theta'_4 = -\frac{3.768(0)}{2.176} = 0$$

These values are shown in parentheses in Fig. 13b, and they are recorded on the diagram as they are computed. Applying Eq. (23a) successively to joints 3, 4, and 6, one obtains

$$\theta'_{5,3} = -\frac{6.001(-1.799) + 1.426(1.00) + 5.333(0)}{1.228} = 7.630,$$

$$\theta'_6 = -\frac{0 + 5.333(-1.799) + 0}{1.702} = 5.637,$$

$$\theta'_{5,6} = -\frac{0 + 5.310(5.637)}{3.544} = -8.422.$$

The discontinuity in slope at joint 5 corresponding to these distortions is

$$\theta'_{5,3} - \theta'_{5,6} = 16.052 ,$$

and the corresponding exciting moment is

$$\bar{M}'_5 = \left[2.362(7.630) + 1.228(-1.799) + 1.052(-8.422) + 3.554(5.637)\right]\frac{EI}{L}$$

$$\bar{M}'_5 = 26.99\,\frac{EI}{L}$$

Next, consider that $\theta_1 = 0$ and that $\theta_2 = 1.00$. The rotations are computed in the same manner and the results are recorded on the diagram below the values for θ'.

The discontinuity in slope and the exciting moment at joint 5 corresponding to these distortions are

$$\theta''_{5,3} - \theta''_{5,6} = 18.242 ,$$

$$\bar{M}''_5 = 33.97\,\frac{EI}{L}$$

The total discontinuity in slope and the total exciting moment are,

respectively,

$$\theta_{5,3} - \theta_{5,6} = 16.052\,\theta_1 + 18.242\,\theta_2 ,$$

$$\frac{\overline{M}_2 L}{EI} = 26.99\,\theta_1 + 33.97\,\theta_2 .$$

The determinant of the coefficients of θ_1 and θ_2 is

$$\Delta_5(\theta_1,\theta_2) = \begin{vmatrix} 16.052 & 18.242 \\ 26.99 & 33.97 \end{vmatrix} = 52.9 .$$

Since this is different from zero, the assumed value of $\lambda_2 = 2.20$ does not correspond to a natural frequency. By repeating this procedure for several values of λ_2 and by evaluating, in each case, the resulting determinant, the curve given in Fig. 14 was obtained. The values of λ_2 corresponding to the natural frequencies of the frame are recorded on the figure. The fundamental natural frequency, in radians per second, is

$$(\omega_n)_1 = \frac{(2.163)^2}{L^2}\sqrt{\frac{EI}{m}} .$$

It can be shown that, for the range of frequencies considered, there are no natural frequencies corresponding to modes of partial vibration.

CONCLUDING REMARKS

A numerical method has been presented for calculating the undamped natural frequencies of flexural vibration of continuous beams on rigid supports and of rigid-jointed, plane frameworks for which the joints do not displace. The members composing the structures have been considered to be of uniform cross section and mass per unit of length.

In this discussion, the effect of permanent axial forces has been neglected. This effect which may be important when the magnitude of the external load on the structure is a sizeable fraction of the critical buckling load, may be taken into account by using modified stiffness and carry-over factors which include the influence of the axial forces. These quantities may conveniently be expressed in terms of the dimensionless parameter λ and the ratio P/P_0, where P is the axial compressive or tensile force and P_0 is the fundamental buckling load of the member considered, assuming its ends to be simply supported. For the limiting case of $\lambda = 0$, the foregoing values of stiffness and carry-over factor reduce to those for an axially loaded bar which does not vibrate. Thus, the problem of investigating the elastic instability of continuous beams and frames becomes a special case of the more general problem of calculating the natural frequencies of flexural vibration of structures in which the component members are subjected to axial forces.(15)

The procedure may be used also to calculate natural frequencies of beams having variable cross section or variable mass per unit of length. A further

extension of the method can be made to the case of continuous beams on an elastic foundation of the Winkler type. To accomplish this, it is only necessary to tabulate the pertinent quantities of stiffness and of the product of the stiffness and the carry-over factor.

An obvious extension of the method is in the analysis of the steady-state forced vibration of continuous beams and frames subjected to harmonically varying forces. For a discussion of this problem the reader is referred to reference (15).

The extension of the method to the case of continuous beams on flexible supports and of simple frames with sidesway has been described elsewhere.[16] The method has been applied also to the case of rectangular plates which are simply supported along two opposite edges and, in one direction, are continuous over rigid supports transverse to the simply supported edges.[15]

ACKNOWLEDGMENT

This paper is based upon a thesis by A. S. Veletsos submitted to the Graduate College of the University of Illinois in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Engineering. The investigation was carried out under the direction of N. M Newmark in the Department of Civil Engineering, and it was supported by the Mechanics Branch of the Office of Naval Research. The tabulated constants were computed on the Electronic Digital Computer of the University of Illinois. The appropriate expressions were coded for machine solution by Mr. A. J. Carlson, Jr., formerly Research Associate in Civil Engineering at the University of Illinois. The writers wish to thank Mr. Carlson for his contribution to this work and also their colleague, Associate Professor W. J. Austin, A. M. ASCE, who read the manuscript and made helpful suggestions.

BIBLIOGRAPHY

1. "The Transverse Vibrations of Beams and the Whirling of Shafts Supported at Intermediate Points," by E. R. Darnley, Philosophical Magazine, Vol. 41, 1921, p. 81.
2. "Vibration Problems in Engineering," by S. Timoshenko, D. Van Nostrand Company, Inc., New York, Third Edition, 1955, pp. 342-345.
3. "Natural Modes of Vibration of Simple Frames," by S. Bennon, Journal of the Franklin Institute, Vol. 243, 1947, pp. 13-19.
4. "Dynamik der Stabwerke," by K. Hohenemser and W. Prager, Julius Springer, Berlin, 1933.
5. "Zue Ermittlung der Grundschwingungszahlen von durchlaufenden Tragern," by W. Mudrak Ingenieur-Archiv, Vol. 7, 1936, pp. 51-55.
6. "Bestimmung der Eigenschwingungszahlen von durchlaufenden Trägern und Rahmen," by W. Mudrak, Zeitschrift für Angewandte Mathematik und Mechanik, Vol. 28, 1948, pp. 258-263.
7. "Vibration Frequencies of Continuous Beams," by E. Saibel, Journal of the Aeronautical Sciences, Vol. 11, 1940, p. 88.

8. "Behavior of Structures Subjected to a Forced Vibration," by C. T. G. Looney, ASCE Proceedings Separate No. 451, Vol. 80, 1954.

9. "On the Fundamental Frequencies of Vibration of Rigid Frames," by E. F. Masur, Proceedings of the First Midwestern Conference on Solid Mechanics, 1953, pp. 89-94.

10. "Die Berechnung der Drechschwingungen," by H. Holzer, J. Springer, Berlin, 1921. Holzer's method is described also in "Mechanical Vibrations," by J. P. Den Hartog, McGraw-Hill Book Company, Inc., N. Y., Third Edition, 1947, pp. 239, 290, or in "Vibration Analysis," by N. O. Myklestad, McGraw-Hill Book Company, Inc., N. Y., 1944, pp. 165-168.

11. "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, Transactions ASCE, Vol. 96, 1932, pp. 1-10.

12. "On Moment Balancing in Structural Dynamics," by R. E. Gaskell, Quarterly of Applied Mathematics, Vol. 1, 1943, pp. 237-249.

13. "Continuous Frames of Reinforced Concrete," by A. H. Cross and N. D. Morgan, John Wiley and Sons, Inc., New York, 1932, p. 119.

14. "A Direct Method of Moment Distribution," by T. Y. Lin, Transactions ASCE, Vol. 102, 1937, pp. 561-571.

15. "A Method for Calculating the Natural Frequencies of Continuous Beams, Frames and Certain Types of Plates," by A. S. Veletsos, Thesis presented to the University of Illinois, Urbana, Illinois, in 1953, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

16. "Determination of the Natural Frequencies of Continuous Beams on Flexible Supports," by A. S. Veletsos and N. M. Newmark, to be published in the Proceedings of the Second U. S. National Congress of Applied Mechanics.

APPENDIX

TABLE I

DIMENSIONLESS COEFFICIENTS OF K AND kK FOR A UNIFORM BAR FIXED AT THE FAR END

The coefficients are given as a function of the dimensionless parameter λ, defined by Eq. (5) in the text.

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$
0	4.0000	2.0000	1.40	3.9632	2.0277	1.80	3.8982	2.0768
0.20	4.0000	2.0000	1.41	3.9621	2.0285	1.81	3.8959	2.0785
0.40	3.9998	2.0002	1.42	3.9610	2.0293	1.82	3.8935	2.0803
0.60	3.9988	2.0009	1.43	3.9599	2.0301	1.83	3.8911	2.0821
0.70	3.9977	2.0017	1.44	3.9587	2.0310	1.84	3.8886	2.0840
0.75	3.9970	2.0023	1.45	3.9576	2.0319	1.85	3.8862	2.0859
0.80	3.9961	2.0029	1.46	3.9564	2.0328	1.86	3.8836	2.0878
0.85	3.9950	2.0037	1.47	3.9552	2.0337	1.87	3.8810	2.0898
0.90	3.9937	2.0047	1.48	3.9539	2.0346	1.88	3.8784	2.0917
0.95	3.9922	2.0058	1.49	3.9527	2.0356	1.89	3.8758	2.0938
1.00	3.9905	2.0072	1.50	3.9514	2.0366	1.90	3.8731	2.0958
1.02	3.9897	2.0077	1.51	3.9500	2.0376	1.91	3.8703	2.0979
1.04	3.9888	2.0084	1.52	3.9487	2.0386	1.92	3.8675	2.1000
1.06	3.9879	2.0090	1.53	3.9473	2.0396	1.93	3.8646	2.1022
1.08	3.9870	2.0097	1.54	3.9459	2.0407	1.94	3.8617	2.1044
1.10	3.9860	2.0105	1.55	3.9445	2.0418	1.95	3.8588	2.1067
1.12	3.9850	2.0113	1.56	3.9430	2.0429	1.96	3.8558	2.1089
1.14	3.9839	2.0121	1.57	3.9415	2.0440	1.97	3.8528	2.1113
1.16	3.9827	2.0130	1.58	3.9400	2.0451	1.98	3.8497	2.1136
1.18	3.9815	2.0139	1.59	3.9385	2.0463	1.99	3.8465	2.1160
1.20	3.9802	2.0149	1.60	3.9369	2.0475	2.00	3.8433	2.1184
1.21	3.9795	2.0154	1.61	3.9353	2.0487	2.01	3.8401	2.1209
1.22	3.9788	2.0159	1.62	3.9336	2.0500	2.02	3.8368	2.1234
1.23	3.9781	2.0164	1.63	3.9319	2.0512	2.03	3.8334	2.1260
1.24	3.9774	2.0170	1.64	3.9302	2.0525	2.04	3.8300	2.1286
1.25	3.9767	2.0175	1.65	3.9285	2.0538	2.05	3.8265	2.1312
1.26	3.9759	2.0181	1.66	3.9267	2.0552	2.06	3.8230	2.1339
1.27	3.9751	2.0187	1.67	3.9249	2.0565	2.07	3.8195	2.1366
1.28	3.9743	2.0193	1.68	3.9231	2.0579	2.08	3.8158	2.1394
1.29	3.9735	2.0199	1.69	3.9212	2.0593	2.09	3.8121	2.1422
1.30	3.9727	2.0205	1.70	3.9193	2.0608	2.10	3.8084	2.1451
1.31	3.9718	2.0212	1.71	3.9174	2.0622	2.11	3.8046	2.1480
1.32	3.9709	2.0218	1.72	3.9154	2.0637	2.12	3.8007	2.1510
1.33	3.9700	2.0225	1.73	3.9134	2.0653	2.13	3.7968	2.1540
1.34	3.9691	2.0232	1.74	3.9113	2.0668	2.14	3.7928	2.1570
1.35	3.9682	2.0239	1.75	3.9092	2.0684	2.15	3.7887	2.1601
1.36	3.9672	2.0246	1.76	3.9071	2.0700	2.16	3.7846	2.1633
1.37	3.9662	2.0254	1.77	3.9049	2.0717	2.17	3.7805	2.1665
1.38	3.9652	2.0261	1.78	3.9027	2.0733	2.18	3.7762	2.1697
1.39	3.9642	2.0269	1.79	3.9005	2.0750	2.19	3.7719	2.1730

TABLE I (Cont'd)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$
2.20	3.7675	2.1764	2.65	3.4865	2.3947	3.10	2.9509	2.8243
2.21	3.7631	2.1798	2.66	3.4780	2.4014	3.11	2.9346	2.8377
2.22	3.7586	2.1832	2.67	3.4693	2.4082	3.12	2.9180	2.8513
2.23	3.7540	2.1868	2.68	3.4605	2.4151	3.13	2.9011	2.8651
2.24	3.7494	2.1903	2.69	3.4516	2.4222	3.14	2.8841	2.8791
2.25	3.7447	2.1939	2.70	3.4426	2.4293	3.15	2.8667	2.8933
2.26	3.7399	2.1976	2.71	3.4334	2.4365	3.16	2.8491	2.9078
2.27	3.7350	2.2014	2.72	3.4241	2.4439	3.17	2.8312	2.9225
2.28	3.7301	2.2052	2.73	3.4147	2.4513	3.18	2.8131	2.9374
2.29	3.7251	2.2090	2.74	3.4051	2.4589	3.19	2.7947	2.9526
2.30	3.7200	2.2129	2.75	3.3954	2.4666	3.20	2.7760	2.9681
2.31	3.7149	2.2169	2.76	3.3855	2.4744	3.21	2.7569	2.9838
2.32	3.7096	2.2209	2.77	3.3755	2.4823	3.22	2.7377	2.9997
2.33	3.7043	2.2250	2.78	3.3654	2.4904	3.23	2.7181	3.0160
2.34	3.6989	2.2292	2.79	3.3551	2.4985	3.24	2.6982	3.0325
2.35	3.6935	2.334	2.80	3.3447	2.5068	3.25	2.6779	3.0493
2.36	3.6879	2.2377	2.81	3.3341	2.5152	3.26	2.6574	3.0663
2.37	3.6823	2.2321	2.82	3.3234	2.5237	3.27	2.6366	3.0837
2.38	3.6766	2.2465	2.83	3.3125	2.5324	3.28	2.6154	3.1013
2.39	3.6708	2.2510	2.84	3.3015	2.5412	3.29	2.5939	3.1193
2.40	3.6649	2.2555	2.85	3.2903	2.5501	3.30	2.5720	3.1375
2.41	3.6589	2.2601	2.86	3.2790	2.5592	3.31	2.5498	3.1561
2.42	3.6528	2.2648	2.87	3.2674	2.5684	3.32	2.5272	3.1750
2.43	3.6467	2.2696	2.88	3.2558	2.5778	3.33	2.5043	3.1942
2.44	3.6405	2.2744	2.89	3.2439	2.5872	3.34	2.4810	3.2138
2.45	3.6341	2.2793	2.90	3.2319	2.5969	3.35	2.4573	3.2337
2.46	3.6277	2.2843	2.91	3.2197	2.6067	3.36	2.4332	3.2539
2.47	3.6212	2.2894	2.92	3.2073	2.6166	3.37	2.4087	3.2746
2.48	3.6146	2.2945	2.93	3.1948	2.6267	3.38	2.3839	3.2955
2.49	3.6079	2.2997	2.94	3.1820	2.6369	3.39	2.3586	3.3169
2.50	3.6011	2.3050	2.95	3.1691	2.6473	3.40	2.3329	3.3386
2.51	3.5942	2.3104	2.96	3.1560	2.6579	3.41	2.3067	3.3607
2.52	3.5872	2.3159	2.97	3.1427	2.6686	3.42	2.2802	3.3833
2.53	3.5801	2.3214	2.98	3.1292	2.6795	3.43	2.2531	3.4062
2.54	3.5729	2.3270	2.99	3.1155	2.6906	3.44	2.2256	3.4296
2.55	3.5656	2.3327	3.00	3.1016	2.7018	3.45	2.1977	3.4534
2.56	3.5582	2.3385	3.01	3.0875	2.7132	3.46	2.1692	3.4776
2.57	3.5507	2.3444	3.02	3.0732	2.7248	3.47	2.1403	3.5023
2.58	3.5430	2.3504	3.03	3.0587	2.7366	3.48	2.1109	3.5274
2.59	3.5353	2.3564	3.04	3.0440	2.7485	3.49	2.0809	3.5531
2.60	3.5275	2.3625	3.05	3.0290	2.7607	3.50	2.0505	3.5792
2.61	3.5195	2.3688	3.06	3.0139	2.7730	3.51	2.0194	3.6058
2.62	3.5114	2.3751	3.07	2.9985	2.7855	3.52	1.9879	3.6329
2.63	3.5033	2.3815	3.08	2.9828	2.7983	3.53	1.9557	3.6605
2.64	3.4949	2.3881	3.09	2.9670	2.8112	3.54	1.9230	3.6887

TABLE I (Cont'd)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$
3.55	1.8897	3.7174	4.05	-1.0838	6.3997	4.55	-20.539	25.402
3.56	1.8558	3.7467	4.06	-1.1891	6.4978	4.56	-22.088	26.941
3.57	1.8213	3.7766	4.07	-1.2977	6.5991	4.57	-23.831	28.672
3.58	1.7861	3.8071	4.08	-1.4096	6.7036	4.58	-25.805	30.635
3.59	1.7503	3.8381	4.09	-1.5249	6.8115	4.59	-28.060	32.879
3.60	1.7138	3.8698	4.10	-1.6439	6.9230	4.60	-30.662	35.469
3.61	1.6766	3.9022	4.11	-1.7668	7.0383	4.61	-33.697	38.492
3.62	1.6387	3.9352	4.12	-1.8937	7.1575	4.62	-37.282	42.066
3.63	1.6001	3.9689	4.13	-2.0247	7.2809	4.63	-41.583	46.355
3.64	1.5607	4.0033	4.14	-2.1603	7.4086	4.64	-46.839	51.599
3.65	1.5206	4.0384	4.15	-2.3004	7.5410	4.65	-53.407	58.155
3.66	1.4797	4.0742	4.16	-2.4455	7.6781	4.66	-61.849	66.585
3.67	1.4380	4.1108	4.17	-2.5957	7.8203	4.67	-73.102	77.826
3.68	1.3954	4.1482	4.18	-2.7513	7.9679	4.68	-88.850	93.561
3.69	1.3520	4.1864	4.19	-2.9127	8.1212	4.69	-112.46	117.16
3.70	1.3078	4.2254	4.20	-3.0802	8.2805	4.70	-151.79	156.48
3.71	1.2626	4.2653	4.21	-3.2540	8.4461	4.71	-230.36	235.04
3.72	1.2165	4.3060	4.22	-3.4347	8.6183	4.72	-465.43	470.09
3.73	1.1694	4.3477	4.23	-3.6225	8.7977	4.73+	$\mp\infty$	$\pm\infty$
2.74	1.1214	4.3902	4.24	-3.8179	8.9847	4.74	480.58	-475.05
3.75	1.0724	4.4338	4.25	-4.0214	9.1797	4.75	242.62	-238.00
3.76	1.0223	4.4783	4.26	-4.2335	9.3832	4.76	163.51	-158.91
3.77	0.97110	4.5239	4.27	-4.4548	9.5958	4.77	124.00	-119.40
3.78	0.91884	4.5705	4.28	-4.6859	9.8180	4.78	100.30	-95.717
3.79	0.86543	4.6182	4.29	-4.9274	10.051	4.79	84.501	-79.934
3.80	0.81084	4.6670	4.30	-5.1801	10.294	4.80	73.219	-68.666
3.81	0.75504	4.7170	4.31	-5.4447	10.550	4.81	64.757	-60.218
3.82	0.69799	4.7682	4.32	-5.7221	10.818	4.82	58.176	-53.650
3.83	0.63965	4.8206	4.33	-6.0133	11.100	4.83	52.909	-48.398
3.84	0.57997	4.8744	4.34	-6.3194	11.397	4.84	48.600	-44.103
3.85	0.51891	4.9294	4.35	-6.6415	11.710	4.85	45.008	-40.525
3.86	0.45642	4.9858	4.36	-6.9809	12.040	4.86	41.967	-37.499
3.87	0.39246	5.0436	4.37	-7.3390	12.389	4.87	39.360	-34.907
3.88	0.32697	5.1029	4.38	-7.7174	12.757	4.88	37.100	-32.661
3.89	0.25990	5.1638	4.39	-8.1180	13.148	4.89	35.122	-30.698
3.90	0.19119	5.2262	4.40	-8.5427	13.563	4.90	33.375	-28.966
3.91	0.12079	5.2902	4.41	-8.9938	14.005	4.91	31.822	-27.428
3.92	0.048634	5.3560	4.42	-9.4739	14.475	4.92	30.431	-26.053
3.93	-0.025344	5.4235	4.43	-9.9858	14.977	4.93	29.179	-24.816
3.94	-0.10121	5.4928	4.44	-10.533	15.514	4.94	28.045	-23.698
3.95	-0.17904	5.5641	4.45	-11.119	16.089	4.95	27.014	-22.682
3.96	-0.25890	5.6373	4.46	-11.748	16.708	4.96	26.072	-21.756
3.97	-0.34087	5.7126	4.47	-12.425	17.375	4.97	25.207	-20.907
3.98	-0.42504	5.7900	4.48	-13.156	18.096	4.98	24.411	-20.127
3.99	-0.51150	5.8696	4.49	-13.948	18.877	4.99	23.676	-19.408
4.00	-0.60034	5.9516	4.50	-14.808	19.727	5.00	22.994	-18.743
4.01	-0.69164	6.0360	4.51	-15.747	20.655	5.01	22.361	-18.126
4.02	-0.78553	6.1229	4.52	-16.774	21.671	5.02	21.771	-17.553
4.03	-0.88210	6.2124	4.53	-17.904	22.790	5.03	21.219	-17.018
4.04	-0.98148	6.3046	4.54	-19.152	24.027	5.04	20.702	-16.518

TABLE I (Cont'd)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$
5.05	20.217	-16.050	5.55	10.660	-7.5873	6.05	7.5233	-6.2553
5.06	19.761	-15.611	5.56	10.576	-7.5312	6.06	7.4711	-6.2504
5.07	19.331	-15.199	5.57	10.494	-7.4768	6.07	7.4190	-6.2461
5.08	18.925	-14.811	5.58	10.413	-7.4242	6.08	7.3670	-6.2425
5.09	18.542	-14.445	5.59	10.334	-7.3732	6.09	7.3151	-6.2396
5.10	18.178	-14.099	5.60	10.256	-7.3239	6.10	7.2632	-6.2373
5.11	17.833	-13.772	5.61	10.179	-7.2761	6.11	7.2113	-6.2356
5.12	17.505	-13.462	5.62	10.103	-7.2298	6.12	7.1595	-6.2346
5.13	17.193	-13.169	5.63	10.029	-7.1850	6.13	7.1077	-6.2342
5.14	16.895	-12.890	5.64	9.9555	-7.1416	6.14	7.0559	-6.2344
5.15	16.612	-12.625	5.65	9.8832	-7.0996	6.15	7.0041	-6.2353
5.16	16.341	-12.373	5.66	9.8120	-7.0590	6.16	6.9522	-6.2368
5.17	16.081	-12.133	5.67	9.7418	-7.0196	6.17	6.9003	-6.2389
5.18	15.833	-11.904	5.68	9.6726	-6.9815	6.18	6.8484	-6.2416
5.19	15.595	-11.685	5.69	9.6042	-6.9447	6.19	6.7964	-6.2450
5.20	15.367	-11.477	5.70	9.5368	-6.9091	6.20	6.7444	-6.2490
5.21	15.148	-11.277	5.71	9.4702	-6.8747	6.21	6.6922	-6.2536
5.22	14.937	-11.086	5.72	9.4044	-6.8414	6.22	6.6400	-6.2589
5.23	14.734	-10.903	5.73	9.3394	-6.8092	6.23	6.5876	-6.2648
5.24	14.538	-10.728	5.74	9.2752	-6.7781	6.24	6.5351	-6.2713
5.25	14.350	-10.560	5.75	9.2116	-6.7481	6.25	6.4825	-6.2785
5.26	14.168	-10.399	5.76	9.1488	-6.7192	6.26	6.4297	-6.2862
5.27	13.992	-10.245	5.77	9.0866	-6.6912	6.27	6.3768	-6.2946
5.28	13.823	-10.096	5.78	9.0251	-6.6643	6.28	6.3237	-6.3037
5.29	13.658	- 9.9534	5.79	8.9642	-6.6384	6.29	6.2704	-6.3134
5.30	13.500	- 9.8160	5.80	8.9038	-6.6134	6.30	6.2169	-6.3237
5.31	13.346	- 9.6839	5.81	8.8441	-6.5893	6.31	6.1631	-6.3347
5.32	13.197	- 9.5567	5.82	8.7848	-6.5662	6.32	6.1092	-6.3463
5.33	13.052	- 9.4342	5.83	8.7261	-6.5439	6.33	6.0550	-6.3585
5.34	12.911	- 9.3162	5.84	8.6678	-6.5226	6.34	6.0006	-6.3715
5.35	12.775	- 9.2024	5.85	8.6101	-6.5021	6.35	5.9459	-6.3850
5.36	12.643	- 9.0926	5.86	8.5527	-6.4824	6.36	5.8909	-6.3993
5.37	12.514	- 8.9868	5.87	8.4958	-6.4636	6.37	5.8356	-6.4142
5.38	12.388	- 8.8846	5.88	8.4393	-6.4457	6.38	5.7800	-6.4298
5.39	12.266	- 8.7860	5.89	8.3832	-6.4285	6.39	5.7241	-6.4461
5.40	12.147	- 8.6908	5.90	8.3275	-6.4121	6.40	5.6679	-6.4630
5.41	12.031	- 8.5988	5.91	8.2721	-6.3966	6.41	5.6113	-6.4807
5.42	11.918	- 8.5099	5.92	8.2170	-6.3818	6.42	5.5544	-6.4990
5.43	11.808	- 8.4240	5.93	8.1622	-6.3677	6.43	5.4970	-6.5181
5.44	11.701	- 8.3409	5.94	8.1077	-6.3544	6.44	5.4393	-6.5379
5.45	11.595	- 8.2606	5.95	8.0536	-6.3419	6.45	5.3812	-6.5584
5.46	11.493	- 8.1829	5.96	7.9996	-6.3301	6.46	5.3226	-6.5796
5.47	11.392	- 8.1078	5.97	7.9459	-6.3190	6.47	5.2636	-6.6016
5.48	11.294	- 8.0351	5.98	7.8925	-6.3086	6.48	5.2042	-6.6243
5.49	11.198	- 7.9647	5.99	7.8392	-6.2989	6.49	5.1442	-6.6478
5.50	11.104	- 7.8966	6.00	7.7862	-6.2899	6.50	5.0838	-6.6720
5.51	11.011	- 7.8307	6.01	7.7333	-6.2817	6.51	5.0229	-6.6971
5.52	10.921	- 7.7668	6.02	7.6806	-6.2741	6.52	4.9615	-6.7229
5.53	10.832	- 7.7051	6.03	7.6280	-6.2671	6.53	4.8995	-6.7496
5.54	10.745	- 7.6452	6.04	7.5756	-6.2609	6.54	4.8369	-6.7770

TABLE I (Cont'd)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$
6.55	4.7738	-6.8053	7.00	0.90203	-9.2964	7.45	-10.007	-18.989
6.56	4.7101	-6.8344	7.01	0.77860	-9.3919	7.46	-10.516	-19.471
6.57	4.6458	-6.8644	7.02	0.65256	-9.4901	7.47	-11.052	-19.980
6.58	4.5808	-6.8953	7.03	0.52383	-9.5911	7.48	-11.616	-20.517
6.59	4.5152	-6.9270	7.04	0.39231	-9.6950	7.49	-12.209	-21.084
6.60	4.4489	-6.9597	7.05	0.25789	-9.8018	7.50	-12.836	-21.683
6.61	4.3818	-6.9933	7.06	0.12045	-9.9117	7.51	-13.498	-22.319
6.62	4.3141	-7.0278	7.07	-0.020117	-10.025	7.52	-14.200	-22.993
6.63	4.2457	-7.0632	7.08	-0.16394	-10.141	7.53	-14.944	-23.710
6.64	4.1764	-7.0997	7.09	-0.31114	-10.261	7.54	-15.734	-24.473
6.65	4.1064	-7.1371	7.10	-0.46186	-10.384	7.55	-16.576	-25.287
6.66	4.0356	-7.1755	7.11	-0.61625	-10.511	7.56	-17.474	-26.158
6.67	3.9639	-7.2150	7.12	-0.77445	-10.642	7.57	-18.434	-27.091
6.68	3.8914	-7.2555	7.13	-0.93663	-10.777	7.58	-19.464	-28.094
6.69	3.8179	-7.2971	7.14	-1.1030	-10.917	7.59	-20.571	-29.173
6.70	3.7436	-7.3398	7.15	-1.2736	-11.060	7.60	-21.764	-30.339
6.71	3.6683	-7.3836	7.16	-1.4488	-11.208	7.61	-23.054	-31.602
6.72	3.5920	-7.4286	7.17	-1.6286	-11.361	7.62	-24.454	-32.974
6.73	3.5147	-7.4747	7.18	-1.8134	-11.519	7.63	-25.977	-34.470
6.74	3.4364	-7.5220	7.19	-2.0034	-11.682	7.64	-27.643	-36.108
6.75	3.3570	-7.5706	7.20	-2.1987	-11.850	7.65	-29.470	-37.908
6.76	3.2765	-7.6204	7.21	-2.3996	-12.024	7.66	-31.486	-39.895
6.77	3.1949	-7.6715	7.22	-2.6065	-12.204	7.67	-33.720	-42.101
6.78	3.1121	-7.7240	7.23	-2.8195	-12.391	7.68	-36.210	-44.564
6.79	3.0281	-7.7778	7.24	-3.0391	-12.583	7.69	-39.003	-47.329
6.80	2.9428	-7.8330	7.25	-3.2655	-12.783	7.70	-42.159	-50.457
6.81	2.8563	-7.8896	7.26	-3.4990	-12.990	7.71	-45.754	-54.024
6.82	2.7684	-7.9476	7.27	-3.7400	-13.204	7.72	-49.886	-58.128
6.83	2.6791	-8.0072	7.28	-3.9890	-13.426	7.73	-54.687	-62.900
6.84	2.5885	-8.0683	7.29	-4.2464	-13.657	7.74	-60.333	-68.518
6.85	2.4963	-8.1310	7.30	-4.5125	-13.896	7.75	-67.070	-75.227
6.86	2.4027	-8.1953	7.31	-4.7879	-14.145	7.76	-75.250	-83.379
6.87	2.3075	-8.2613	7.32	-5.0732	-14.403	7.77	-85.392	-93.492
6.88	2.2107	-8.3290	7.33	-5.3688	-14.672	7.78	-98.301	-106.37
6.89	2.1123	-8.3985	7.34	-5.6754	-14.952	7.79	-115.29	-123.33
6.90	2.0121	-8.4698	7.35	-5.9936	-15.243	7.80	-138.66	-146.67
6.91	1.9101	-8.5430	7.36	-6.3242	-15.547	7.81	-172.84	-180.82
6.92	1.8064	-8.6181	7.37	-6.6679	-15.864	7.82	-227.60	-235.55
6.93	1.7007	-8.6952	7.38	-7.0256	-16.195	7.83	-329.54	-337.46
6.94	1.5930	-8.7744	7.39	-7.3981	-16.541	7.84	-585.85	-593.75
6.95	1.4834	-8.8556	7.40	-7.7864	-16.902	7.85	-2441.7	-2449.6
6.96	1.3716	-8.9391	7.41	-8.1916	-17.281	7.86	1164.5	1156.7
6.97	1.2577	-9.0249	7.42	-8.6149	-17.677			
6.98	1.1415	-9.1129	7.43	-9.0576	-18.093			
6.99	1.0230	-9.2034	7.44	-9.5210	-18.530			

AMERICAN SOCIETY OF CIVIL ENGINEERS

OFFICERS FOR 1955

PRESIDENT

WILLIAM ROY GLIDDEN

VICE-PRESIDENTS

Term expires October, 1955:
ENOCH R. NEEDLES
MASON G. LOCKWOOD

Term expires October, 1956:
FRANK L. WEAVER
LOUIS R. HOWSON

DIRECTORS

Term expires October, 1955:
CHARLES B. MOLINEAUX
MERCEL J. SHELTON
A. A. K. BOOTH
CARL G. PAULSEN
LLOYD D. KNAPP
GLENN W. HOLCOMB
FRANCIS M. DAWSON

Term expires October, 1956:
WILLIAM S. LaLONDE, JR.
OLIVER W. HARTWELL
THOMAS C. SHEDD
SAMUEL B. MORRIS
ERNEST W. CARLTON
RAYMOND F. DAWSON

Term expires October, 1957:
JEWELL M. GARRELTS
FREDERICK H. PAULSON
GEORGE S. RICHARDSON
DON M. CORBETT
GRAHAM P. WILLOUGHBY
LAWRENCE A. ELSENER

PAST-PRESIDENTS

Members of the Board

WALTER L. HUBER
DANIEL V. TERRELL

EXECUTIVE SECRETARY
WILLIAM H. WISELY

TREASURER
CHARLES E. TROUT

ASSISTANT SECRETARY
E. L. CHANDLER

ASSISTANT TREASURER
CARLTON S. PROCTOR

PROCEEDINGS OF THE SOCIETY

HAROLD T. LARSEN
Manager of Technical Publications

DEFOREST A. MATTESON, JR.
Editor of Technical Publications

PAUL A. PARISI
Assoc. Editor of Technical Publications

COMMITTEE ON PUBLICATIONS

SAMUEL B. MORRIS, *Chairman*

JEWELL M. GARRELTS, *Vice-Chairman*

GLENN W. HOLCOMB
OLIVER W. HARTWELL
ERNEST W. CARLTON
DON M. CORBETT